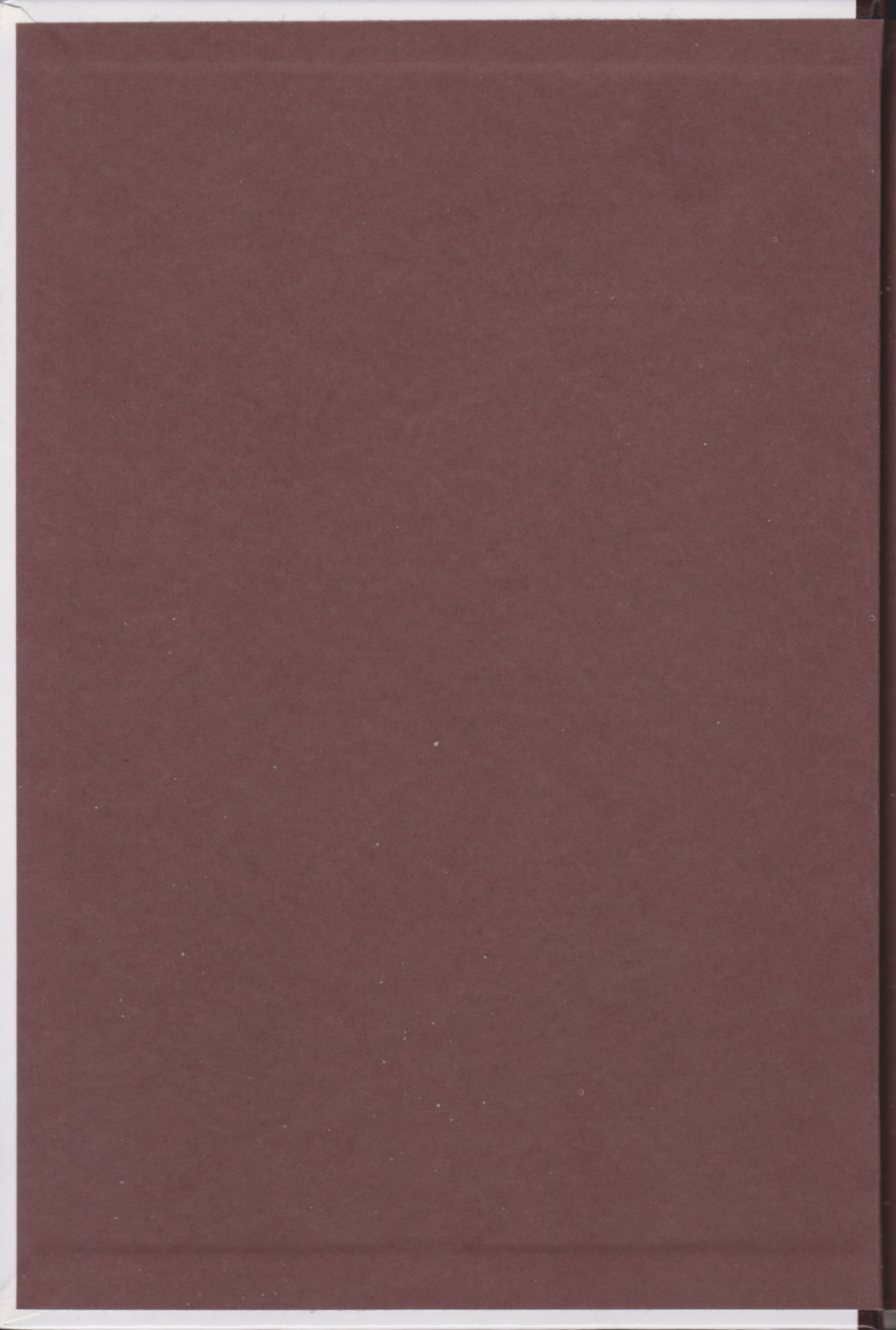


Pythagoras' Theorem

The sacred geometry of triangles



Everything is mathematical



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Claudi Alsina

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*"Geometry has two great treasures: One is the
Theorem of Pythagoras, and the other the division of
a line into extreme and mean ratio. The first we can
compare to a measure of gold, the second we may call
a precious jewel."*

Johannes Kepler (1571-1630)"

Preface

Human beings throughout the world share many things, cultural features unique to our entire species that go far beyond the differences we sometimes endeavour to find between us. A good example of this is Pythagoras' theorem, a fundamental mathematical result, which all cultures have discovered or developed in one way or another.

Maybe you have asked yourself why you they made you study Pythagoras' theorem. Where does it come from? What use is it? What can you get from it? This book provides all the answers to these simple questions and also reveals many other interesting subjects related to this most celebrated theorem, including numerical roots.

At some point of your education you will have undoubtedly come across the concepts of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{a^2 + b^2}$ and perhaps $\sqrt{-1}$ too. It is very possible that you tackled them with the help of a calculator, and this may have prevented you from paying them proper attention. After all, we are talking about concepts that are enormously important to geometry and algebra and which, surprising as it may seem, in their day aroused genuine hatred and passion alike.

A book on Pythagoras' theorem cannot begin without a look at the dawn of mathematics itself. It started with the anonymous efforts of many Babylonians and Egyptians, who prepared the way for the achievements of the scholars of the Classical Greek period. Among them Pythagoras is undoubtedly history's first mathematical superstar, albeit one shrouded in legend, part mystic and part cult leader. Beyond the myth, Pythagoras continues to be one of the most influential names in the story of human knowledge. The theorem that carries his name has a prominent place in the treatises of classic geometry. Revisiting it offers the perfect opportunity to enjoy the beauty of some of its visual demonstrations, as simple as they are ingenious. It is also a starting point for exploring some of mathematic's conundrums that have roused the interest of experts and laymen for centuries and which are much more present in our daily lives than most of us imagine.

But Pythagoras' theorem is much more than a ratio of the sides of a right-angled triangle. Its simple formulation is also a door from which you can immerse yourself in $\sqrt{2}$, its history, properties and mysteries. It is not in vain that $\sqrt{2}$ was history's first irrational number, the first value that was not a fraction. Its discovery altered the early landscape of mathematics forever. Through this and other square roots, you will dive into some incredible constructions such as the Spiral of Theodorus, and discover more about the celebrated 'golden ratio' – and its other precious metal

relatives – which forms a direct link between mathematics, on the one hand, and nature and the arts, on the other.

The applications of the theorem are diverse and all pervading. With it, polygons can be squared, similar figures added and Hippocrates' moon problem solved. It also provides a scientific basis for producing perspective in artworks. As if all this were not enough, the theorem throws light on Pythagorean triples and Fermat's last theorem, which had to wait three hundred years to be unlocked, and whose powers extend beyond the world of triangles and other flat shapes and makes surprising appearances in three dimensional space.

Chapter 1

Pythagoras and the dawn of mathematics

The foundation of mathematics as a theoretical science is attributed to Pythagoras and his followers in the 5th century B.C. It was Pythagoras who understood that the certainty of propositions needed to be proved before they could be used in further logical demonstrations, and he did so even before Euclid, the great compiler and reorganiser of classical mathematics. To do this, the master took a fundamental element of philosophy, logic, and applied it to mathematics in a manner so essential that it now seems that philosophy has taken it from mathematics. For the Pythagoreans, mathematics was not a mere scientific attitude; it was the explanation of the world, the instrument for understanding nature and the path to perfection. The influence of his philosophy has permeated Western culture ever since.

It is easy to understand the success of Pythagoreanism. Pythagoras was a contemporary of Buddha (563 or 623 B.C.–483 or 543 B.C.), Confucius (551–479 B.C.) and Lao-Tse (uncertain, between 6th and 4th centuries B.C.), who were also founders of their own philosophies – philosophies in which it is difficult to pinpoint where religion ends and philosophy begins. The Pythagorean doctrine was a perfect synthesis between mysticism and rational thought, a mixture of science and religion that proposed a genuine way of life. Therein lies its powerful force as a cultural movement. The combination of mathematics and theology that began with Pythagoras characterised the religious philosophy of ancient Greece, the Middle Ages, and is felt to the present day.

When Pythagoras, at forty years of age, begun to explain his doctrine in Samos, his native island, he had just returned from a long journey. He had been in Egypt and Babylonia (some suggest he went as far as India). In those lands, Pythagoras soaked up the oriental religious spirit but also absorbed all he could about mathematics and astronomy. When he returned, he was imbued with all that knowledge and set about combining it with the heritage of his own culture. Many elements in the Pythagorean doctrine were inherited from earlier scholars, although they were enhanced in many cases by the master. Not least among them is Pythagoras'

most celebrated discovery about right-angled triangles: The sum of the squares of the edges of a right-angled triangle is equal to the square of the hypotenuse. This is internationally recognised as 'Pythagoras' theorem', but its history goes back to even earlier times.

The first civilisations

The Fertile Crescent extends from the Euphrates and Tigris Rivers to the mountains of Lebanon. It is currently occupied mainly by Iraq, and in the remote past, in the very same place, some of the most splendid civilisations of all time flourished there: the Sumerians, the Akkadian empire and, finally, the Babylonians, whom we group together under the umbrella term of Mesopotamia (meaning 'the land between two rivers'). In the last two centuries, archaeologists have found hundreds of thousands of clay tablets with cuneiform characters (wedge-shaped marks named with the word Latin *cuneus*, meaning 'wedge'). These were engraved with a stylus on fresh clay, which was later dried in the sun or in an oven. These tablets bear witness to the fact that these peoples developed commerce and architecture, made detailed astronomical observations and, under the Babylonian government of Hammurabi, drafted the first legal code in history, among many other fundamental milestones for humanity. Until now only a small part of this legacy has been studied. The vast majority of the Babylonian tablets are in the basements of museums around the world waiting to be decoded. They will doubtless further surprise the international community by revealing the grand achievements of these cultures, which have been buried in the sand for nearly 3,500 years.

So far 500,000 tablets have been found, of which 300 have mathematical content. A tablet that has received great attention is the so-called Plimpton 322.



The Plimpton 322 tablet, currently at Columbia University, New York.

This title indicates that the tablet is number 322 in the U.S. journalist George Arthur Plimpton's collection, which was transferred to Columbia University in 1936. It belongs to the ancient period of the Hammurabi dynasty (from approximately 1800 to 1600 B.C.) and shows a table with four columns full of cuneiform characters, which appear to be numbers written in the Babylonian base 60 number system.

At a glance, the jumbled columns full of numbers could look like a record of commercial transactions, but detailed studies have revealed something quite different and truly extraordinary. The tablet is a list of Pythagorean triples, which are combinations of positive integers (a, b, c) , such that $a^2 + b^2 = c^2$. Here are some examples of triples: $(3, 4, 5)$, $(5, 12, 13)$ and $(8, 15, 17)$. According to Pythagoras' theorem, each of these triples represents the lengths of the sides of a right-angled triangle. The Plimpton 322 tablet shows nothing less than the fact that the Babylonians were familiar with geometry and algebraic procedures.

Since its discovery, the content of the tablet has been given several interpretations. Some experts in Babylonian mathematics, such as Swede Jören Friberg, have pointed to a possible algebraic method for calculating the Pythagorean triples. Other authors have argued that a geometric model lies beneath the figures. It was the Austrian mathematician Otto Neugebauer who proved that this was indeed a list of Pythagorean triples and led his British colleague Christopher Zeeman to introduce the popular expression that now defines the Plimpton 322 tablet as "the oldest preserved document on number theory". In 2001 science historian Eleonor Robson highlighted the tablet's educational function, suggesting that this was a supporting document for teachers used to tackle problems on right-angled triangles and square and reciprocal numbers.

The discovery was a great surprise and turned our understanding of the history of maths on its head. How did the Babylonians find the Pythagorean triples? Why were they interested in these numbers? In order to create what could be considered the first trigonometric table in history, they must have known an algorithm for the creation of triples for which no records survived until it was set out by Euclid in his *Elements* 1,500 years later. Thus, the Babylonians not only must have known Pythagoras' theorem, but they must have had the rudiments of number theory and also sufficient computation ability to be able to take it from theory and put it into practice – all of this thousands of years before ancient Greece produced its first great mathematician. Should we start attributing the discoveries traditionally associated with Pythagoras to the Babylonians instead?

THE FIRST TRIGONOMETRIC TABLE?

The table shows the decoded Babylonian numbers, allowing a detailed mathematical analysis that contains some major surprises. Even with the cuneiform characters decoded, some explanation is needed to understand the table. It is evident that column IV simply gives order to the rows. Columns II and III give the values of the hypotenuse and of one side of a right-angled triangle in base 60. In the last column, named *l*, the values of the remaining side are given. The first column is slightly mysterious, since it gives the squares of the *d* ratios divided by *l*, which could today be described as the square of a trigonometric function.

As an example, we are going to modify row 1: in it, using our numbers, side *b* = 119, hypotenuse *d* = 169, making the other side 120.

I.	II. <i>b</i>	III. <i>d</i>	IV.	<i>l</i>
(1) 59 00 15	1 59	2 49	1	2 00
(1) 56 56 58 14 50 06 15	56 07	3 12 1 [1 20 25]	2	57 36
(1) 55 07 41 15 33 45	1 16 41	1 50 49	3	1 20 00
(1) 53 10 29 32 52 16	3 31 49	5 09 01	4	3 45 00
(1) 48 54 01 40	1 05	1 37	5	1 12
(1) 47 06 41 40	5 19	8 01	6	6 00
(1) 43 11 56 28 26 40	38 11	59 01	7	45 00
(1) 41 33 59 03 45	13 19	20 49	8	16 00
(1) 38 33 36 36	9 01 [8 01]	12 49	9	10
(1) 35 10 02 28 27 24 26 40	1 22 41	2 16 01	10	1 48 00
(1) 33 45	45	1 15	11	1 00
(1) 29 21 54 02 15	27 59	48 49	12	40 00
(1) 27 00 03 45	7 12 1 [2 41]	4 49	13	4 00
(1) 25 48 51 35 06 40	29 31	53 49	14	45 00
(1) 23 13 46 40	56	53 [1 46]	15	1 30

Building the Great Pyramid

Throughout the Nile valley in Egypt, more than 1,200 kilometres to the southwest of Mesopotamia, another most intriguing ancient civilisation went about its business. For three thousand years, from 3500 B.C. until the Classical Greek era, the Mesopotamians and the Egyptians coexisted in relative peace. Both peoples developed sophisticated writing systems, observed the heavens with great diligence and meticulously recorded their military victories and commercial activities creating an invaluable cultural legacy. However, while the Babylonians made their records on their famous clay tablets, which are difficult to destroy, the Egyptians used a very fragile medium, papyrus. Dozens of centuries later, only the dry climate of the desert has saved all of these documents from disintegration. Nevertheless, and unbelievable as it sounds considering the sheer amount of ancient Egyptian buildings that survive, we know more about the extent of Mesopotamian knowledge than we do about Egypt. Everything known about the ancient Egyptians comes from there main sources: from the objects that have been found in their royal tombs, from a handful of rolls of papyrus that have been miraculously preserved and from the hieroglyphic inscriptions that adorn their temples and monuments.

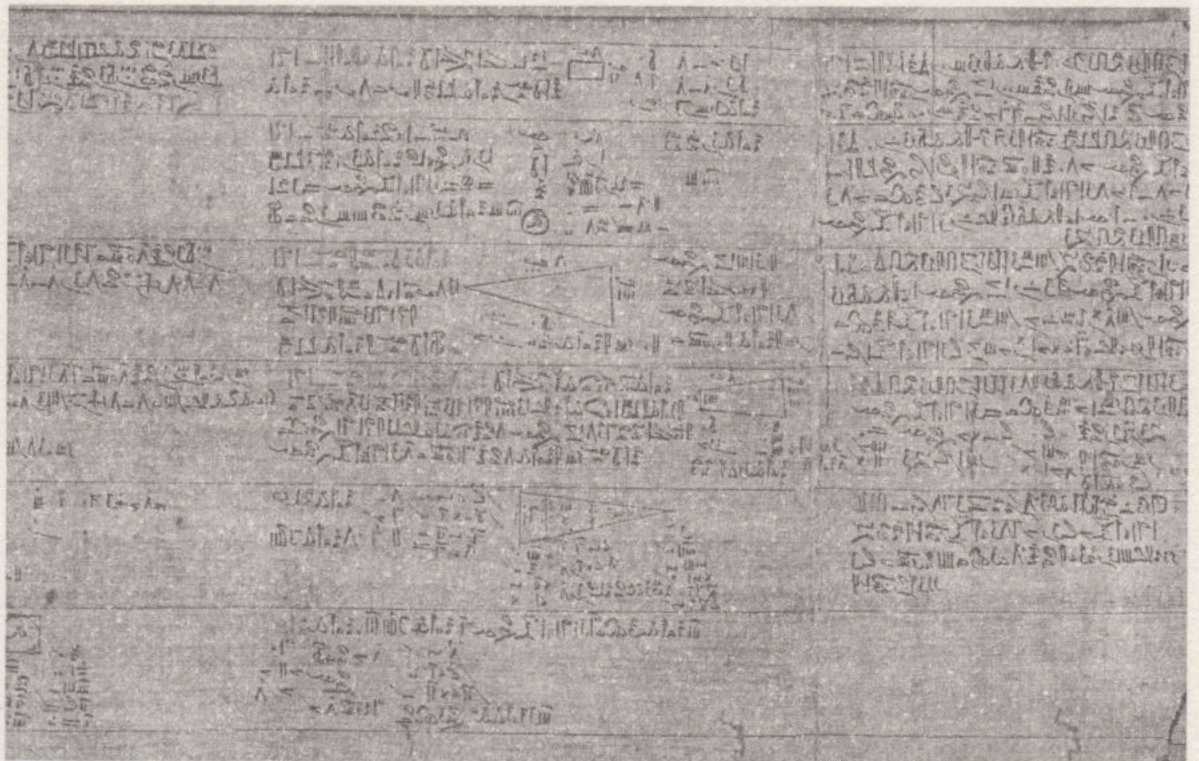
Tonnes of ink has been spilt over the Pyramids of Giza, the most admired of ancient Egyptian constructions, but most of this literature is more fiction than scientific reality. The pyramids attract myriads of enthusiasts who reveal hidden connections to almost anything, from the numerical values of π and the golden ratio to the alignment of the planets and the stars.

It seems evident that to raise a monument as enormous as the Great Pyramid of Cheops must require a relatively advanced knowledge of mathematics, and it seems reasonable to think that knowledge must have included Pythagoras' theorem. But has this been proven? The best evidence about the state of mathematics in ancient Egypt is the Rhind papyrus, an anthology of 87 rudimentary arithmetic, geometric and algebraic problems discovered by the Scottish Egyptologist Alexander Henry Rhind in 1858. Luckily, the paper has survived several millennia in excellent conditions, and it forms the world's oldest mathematics text book, and remains more or less completely intact. It was written around 1650 B.C. by a scribe named A'h-mose, known in the western world as Ahmes. He is not the author of the work, as Ahmes himself explains; it is a copy of an older document, from 1800 B.C. Its 87 problems contain step-by-step solutions and some are even accompanied by drawings. It was most probably a manual for the school of royal scribes, the group that carried

out all the reading, writing and arithmetic work. The papyrus itself begins with a celebrated if rather ambitious affirmation:

“Exact calculation to gain greater knowledge of all existing things and all the dark secrets and mysteries.”

The 87 problems in the Rhind papyrus are about numbers and their functions, calculating fractions, linear equations, progressions and distributions. Twenty of them are of a geometric nature and present problems such as finding the volume of a cylindrical granary or the area of a field of given dimensions. These were very important problems for the Egyptians, who depended on the annual flooding of the Nile for their survival, marking out precise field boundaries after each inundation. Five of the problems are specifically about the pyramids, and they make no reference, not even indirectly, to Pythagoras’ theorem. A concern that appears repeatedly is the calculation of the slope of the side of a pyramid, fundamental for the builders, who



The Rhind Mathematical Papyrus, which dates from approximately 1650 B.C., is the oldest mathematics text book and survives almost intact.

THE RHIND MATHEMATICAL PAPYRUS: A CLOSE-UP

In this close-up of the Rhind Mathematical Papyrus we can see familiar mathematical figures. The triangle shown here belongs to problem 51 of the papyrus. Its object is to find the area of a triangle with a height is 10 *khet*s and a base of 4 *khet*s. A *khet*, or 'rod', was a measurement of 100 cubits, and the Egyptian royal cubit is equal to 52.3 cm (it was divided into 7 palm spans).



Thus, the measurements of the triangle would be: 523 m high by a 209 m base. Ahmes' solution shows that it is an isosceles triangle, divided into two by the central axis, therefore it can be used to form a rectangle with an equal area.

had to make sure that all four sides of the pyramids had a uniform slope. But there is no sign of Pythagoras' theorem.

If the Rhind papyrus represents the summary of mathematics that a member of the educated elite – a scribe, an architect or a tax collector – must have known, does the absence of any reference to the Theorem of Pythagoras mean that the Egyptians did not know it? It is well known that, as a practical resource for verifying right angles, they used a set of ropes with 3, 4 and 5 knots (the simplest Pythagorean triple) placed at regular intervals. However, there is no evidence that this practical knowledge led them to formalise the ideas behind the right-angled triangle and a general theorem that linked the lengths of its sides. If the Egyptians used some kind of theorem like that of Pythagoras to build the pyramids, archaeologists are yet to have dug up the document that proves it.

Together with the Rhind papyrus, the most important mathematical document of ancient Egypt is the famous Moscow papyrus, dated 1890 B.C., currently kept in the Moscow Museum of Fine Arts from which it gets its name. The document's shape is peculiar – it is 5 metres long but only 8 centimetres wide. In that narrow space there are 25 mathematical problems, some of them too worn to be interpreted, presented by an unknown scribe who was less meticulous than Ahmes. The Moscow papyrus contains calculations on the volume of a truncated pyramid, but, above all, problem 14 is noteworthy as it presents, for the first time, the exact formula for the volume of a frustum of a square pyramid. However, once again, there is no Pythagorean ratio in evidence. (A frustum is the remainder of a cone or pyramid that has had the point sliced off parallel to the base.)

Only in the Berlin papyrus, a series of medical, literary and mathematical documents from the Middle Kingdom, can any trace of the theory be found. One section includes uses a simultaneous equation to resolve the two unknowns in the following problem:

“You are told the area of a square of 100 square cubits is equal to that of two smaller squares. The side of one square is $1/2 + 1/4$ of the other. Find the sides of the squares.”

In current algebraic language, the problem is the equivalent to solving the simultaneous equation

$$\begin{aligned}x^2 + y^2 &= 100, \\ y &= (1/2 + 1/4)x,\end{aligned}$$

which involves calculating a square root, exactly as is done on the papyrus. This may now be seen as a Pythagorean approach, but rather than suggesting a specific knowledge of Pythagoras’ theorem, this could just be a lesson about quadratic equations.

Greek scientific thought

In the 6th century B.C. Mesopotamia and Egypt stopped being the epicentre of Western thought, which had now shifted into the Greek world. Through their influence over the later Roman Empire, Greek thought would dominate Europe, remaining largely unquestioned for at least the next 16 centuries. It took a long time to gain a more objective understanding of the actual importance of this Greek heritage compared to the contributions of other ancient civilisations. It all began in 1799 with the discovery of the Rosetta Stone in Egypt by one of Napoleon’s soldiers. This was deciphered 25 years later by Jean-François de Champollion and Thomas Young, allowing many more ancient texts to be read, not just those in Greek and Latin. They revealed other approaches to science and philosophy offering a better understanding of the intellectual achievements of ancient civilisations. We now know that contributions from many Greek sources were not always the result of original thought, although that does not make them any less important.

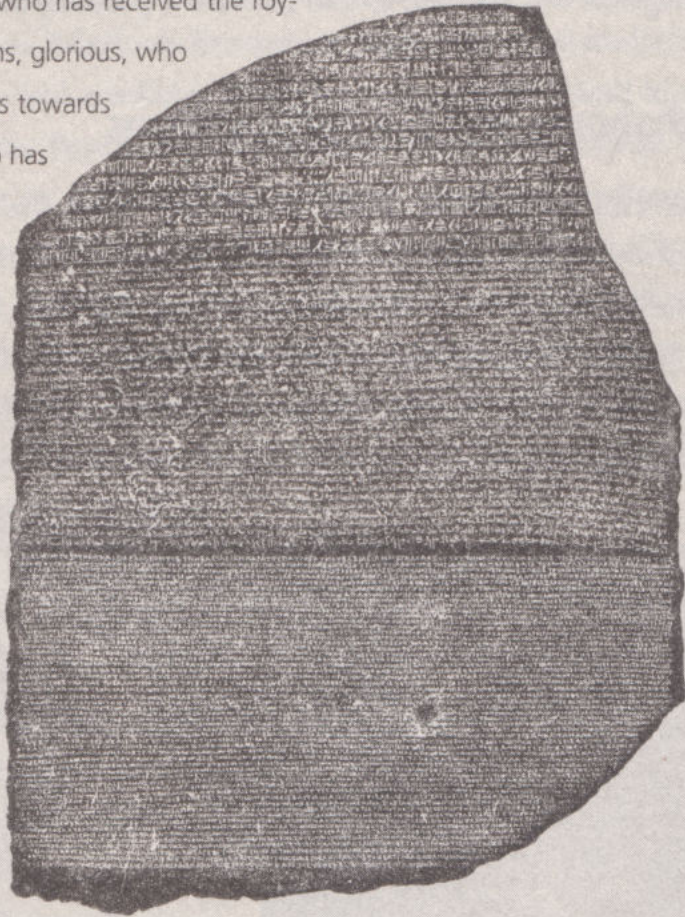
The Ionian Greeks of the 6th century B.C., who inhabited Asia Minor (now Turkey), began to separate rational knowledge from mythological belief. This gave

THE ROSETTA STONE

The Rosetta Stone is a black granite stela that bears the same text, a decree from Pharaoh Ptolemy V, written in Greek and Egyptian, with three types of writing: hieroglyphics, demotic Egyptian and ancient Greek. Therefore it is also a comparative table of Greek and Egyptian, which allowed Champollion and Young to decipher the hieroglyphic writing for the first time.

The text displays a sentence from Ptolemy V revoking taxes and ordering that the decree be published in the language of the gods, hieroglyphics, and the language of the people, demotic. The latest expert studies of demotic have demonstrated that the original inscription was written in Greek and then translated to demotic, with a few corrections. The translation of the first paragraph in Greek of the Rosetta stone explains the following:

"In the reign of the young one – who has received the royalty from his father – lord of crowns, glorious, who has established Egypt, and is pious towards the gods, superior to his foes, who has restored the civilised life of men, lord of the Thirty Years' Feasts, even as Hephaistos the Great; a king, like the Sun, the great king of the upper and lower regions; offspring of the Gods Philopatores, one whom Hephaistos has approved, to whom the Sun has given the victory, the living image of Zeus, son of the Sun, Ptolemy living-for-ever beloved of Ptah; in the ninth year, when Aëtus, son of Aëtus, was priest of Alexander, and the gods Soteres, and the gods Adelphoi, and the gods



Euergetai, and the gods Philopatores and the god Epiphanes Eucharistos; Pyrrha daughter of Philinos being Athlophoros of Berenike Euergetis; Areia daughter of Diogenes being Kanephoros of Arsinoe Philadelphos; Irene, daughter of Ptolemy being Priestess of Arsinoe Philopator; the fourth of the month of Xandikos, according to the Egyptians the 18th Mekhir."

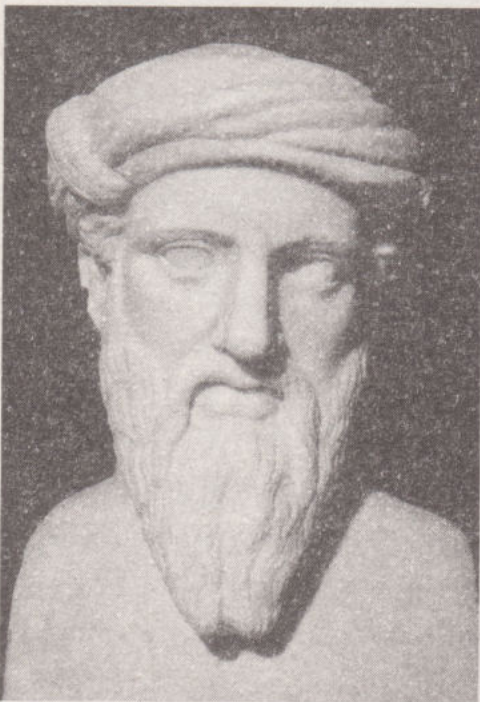
the Greeks a tool that other cultures had never developed – the power to theorise. The Babylonian and Egyptian ethos was less organised, based on chance discovery, experiments and intuition. To them mathematics was a tool, a way of solving practical problems, rather than a means to reveal general laws. By contrast, the Greeks considered the pursuit of knowledge through science to be an end in itself.

The Greeks borrowed a lot of geometry from the Egyptians. Also, Greek merchants had been learning the practical mathematics of Mesopotamia and India. But it was to be in Greece where all this knowledge, collected from here and there, synergised into something new. Even the word *mathematics* has Greek origins; *mathema* is ‘that which is taught’, meaning ‘all forms of knowledge’. The Pythagoreans were the first to use the term *mathematikoi*, ‘mathematics’, and their great master Pythagoras was the first to call himself a ‘philosopher’ – lover of knowledge.

Pythagoras and the Pythagoreans

Countries around the world frequently assess the scientific knowledge of their populations, and one question always appears: “Do you know the name of any mathematicians?” For the overwhelming majority, the reply is “Pythagoras”. Very few people know of any others. Pythagoras is considered the first recorded mathematician and his famous theorem continues to provide the foundations of basic mathematical teaching in most schools. But who was Pythagoras? The truth is that

we have very little idea.



Pythagoras is one of history’s most mysterious figures. The scarce information available to us is rather suspect, since it seems more like fiction than reality. The historians who wrote it lived hundreds of years after the period in question. Tradition states that the master was born around 570 B.C. on the island of Samos, in the Aegean Sea, off the coast of what is now Turkey. According to some authors, he was the son of Apollo, but others attribute him a more banal father, Mnesarchus, a wealthy is-

An ancient Greek marble bust of Pythagoras.

lander. Samos was not very far from the coastal city of Miletus, home to the famous philosopher Thales. To many, Thales is the founding father of science and the first of the extensive line of Greek thinkers that shaped the intellectual world for the next millennium. We could assume, although we cannot be sure, that the young Pythagoras met him, and that Thales encouraged his passion for mathematics and philosophy. The tradition of demonstrating results in mathematics as practised by Pythagoras was first suggested by Thales.

The journeys of Pythagoras' youth are stuff of legend and really only serve to illustrate the story behind the formative years of this genius, since we cannot be sure where he went nor what he did there. It is said that he travelled to the ancient world's greatest centres of civilisation, among them Egypt and Persia, where he absorbed their literature, their religion, their philosophy and their mathematics. It seems probable that he settled in Babylonia for twenty years, where he studied and taught astronomy, mathematics and astrology. When he returned, he found himself under the reign of Polycrates the Tyrant, and so sometime around 530 B.C. he once again quit Samos. It is said that he then travelled to Egypt, where he continued to study. Whatever may have happened, we know for certain that he settled in Croton, in the south of Italy. There he founded a society of disciples, a school of thought that acquired notable prestige in the city and exercised enormous influence on the following generations of thinkers and scholars. But, in the end, it seems that the citizens of Croton turned against Pythagoras, and he had to leave for Metapontum.



A map of the eastern Mediterranean region, showing where Pythagoras is thought to have lived.

Under the guidance of their master, the Pythagoreans dedicated themselves to study each and every one of the intellectual disciplines of the time, above all philosophy, mathematics and astronomy. But theirs was more than just a school of thought. The Pythagoreans were a community adhering to a series of strict rules; a

society cloaked in secrecy that came to resemble a religious cult. The community was devoted to the exploration of the mysteries of numbers and was hugely conservative and authoritarian. It was not a fraternity or an alliance, as it has sometimes been described, but a collection of distinct families. It could be considered the model of the ideal society as described by Plato in *The Republic*. As far as we know, all property was communal, and the members had to exercise self-control and decorum. The main leaders practised celibacy. It seems that women were accepted as equals in all group activities, a genuine rarity in the ancient world. The fact that women were teaching mathematics suggests that they were being educated to unprecedented levels for that era.

Some of the classic legends of the Pythagorean sect are about the death of Pythagoras. It seems that the initiation tests for the community were very severe and lasted for years. It is said that some of the rejected men, resentful for not having been accepted due to failing the tests, set fire to a house with the master and other Pythagoreans inside. The destruction of this house is said to be the beginning of the end of the sect, which was forced to disperse. Some versions of the story go a step further: as he fled, the enemies chased Pythagoras to a bean field. The Pythagoreans revered beans and were said to never eat or even touch them. And so, unable to carry on without breaking his own rules, Pythagoras chose to die rather than trample over some beans.

After his death, Pythagoras became a mystical figure, a performer of miracles and attributed with magic powers that make him one of the most interesting and disconcerting characters in history. However, the strict vow of silence taken by his follower never to reveal their discoveries had unfortunate consequences. It prevented their research from reaching the wider public and aroused great suspicion of what they were up to. And today it means that everything we know about the Pythagoreans is from second-hand references from later generations of writers, whose normal objective was to praise their hero Pythagoras.

On top of this lack of original sources, the Pythagoreans followed the oriental tradition of the oral transmission of knowledge. Written material is very scarce. Egyptian papyrus reached Greece in around 650 B.C., and in Pythagoras' day there were few in circulation. Parchment was still undiscovered and a clay tablet was not the most suitable media for writing long philosophical discourses. Thus, the knowledge was passed from one generation to the next in the form of words whispered by the fire or proclaimed to the heavens. Also, out of respect to their master, the community's discoveries were nearly always attributed to Pythagoras. So we have

no genuine evidence of the discoveries, nor do we know what if anything Pythagoras actually discovered himself.

The Golden Verses

The Golden Verses is the list of rules and traditions used to regulate the community. Some of them coincide almost verbatim with the Christian commandments, which some suggest they inspired. They are very practical rules, neither strange nor ridiculous. Above all, they are moral and seek to regulate the community, favour prosperity and encourage people to study. They were written in the form of aphorisms, which the Pythagoreans read in the morning as a declaration of intentions, and at night for meditation. Horace, Seneca and Cicero followed this Pythagorean ritual (and recommended others do so). It is possible that the verses come from Pythagoras' speeches and opinions, but most scholars agree that they are more likely a compilation of contributions from several Pythagoreans from before the 3rd century B.C.

Irrespective of their origin, anyone living at any time would be happy to subscribe to most of the Golden Verses, such as that which proclaims something as simple as "make him thy friend who distinguishes himself by his virtue", or well intentioned as "be kind with your words and useful with your work". Others, on the other hand, are more dubious, such as "always abstain from what you do not know". In any case, the wisdom that they contain is unquestionable and still capable of bringing enlightenment. For example, "Learn everything necessary so that your life be easier, but do not neglect the health of your body. Do this by trying to find the just measure. And understand the just measure as that which does not cause pain".

The first four Golden Verses are surprisingly similar to the first four commandments passed down by Moses.

THE GOLDEN VERSES	THE JEWISH-CHRISTIAN COMMANDMENTS
Honour the immortal gods according to that established by law.	1. You shall love God above all things.
Reverence thy oath, and then the illustrious heroes	2. You shall not take the name of God in vain.
And the same for buried geniuses, according to the traditional rites.	3. Remember the Sabbath and keep it holy.
Honour your father and your mother, as well as your relatives.	4. You shall honour your father and your mother.

Time has obscured the Golden Verses. People are more likely to know about other Pythagorean traditions, which are more like a mockery, or at least, a defamation of the contemplative sect. Here is a sample:

- Abstain from beans.
- Do not pick up anything which has fallen.
- Do not touch a white cockerel.
- Do not break bread.
- Do not walk a beam.
- Do not pick up a wreath.
- Do not eat heart.
- Do not walk on the roads.
- Do not let swallows make their nests in the roof of your own house.
- When the cooking pot is removed from the fire, do not leave its mark in the ash, but erase it.
- Do not look into a mirror beside a light.
- When taking off the sheets, roll them up until the imprint of your body disappears.

Philosophy and science of the Pythagoreans

The Pythagoreans subscribed to five basic ideas that have had a powerful influence on mankind's thinking ever since.

1. The universe was created, and continues to exist, on the basis of a divine plan. The ultimate reality is not material but spiritual; it consists of the ideas of number and form. Ideas are divine concepts that are superior to matter and independent of it.
2. God created souls as spiritual entities. The soul is a "self-moving number" which inhabits a body for a limited amount of time and then passes to another body, be it that of a human or another animal. Souls are eternal and only purification, through strict morals and an intellectual life, can liberate them from this endless cycle and elevate them to union with God. This is the concept of reincarnation of souls, the metempsychosis, and is the idea that justifies the vegetarian diet of the Pythagoreans.

3. There is an inner harmony and order to the universe, resulting from the union of opposites. There are ten pairs of fundamental opposites interacting in a creative way: even-odd, masculine-feminine, good-bad, wet-dry, right-left, rest-motion, cold-hot, light-darkness, straight-curved, limited-unlimited.
4. In human relationships, friendship and modesty are the main fundamentals. Men and women should live an ascetic life in a cooperative group devoted to the rearing of children in harmony with the divine plan.
5. The divine ideas, which created and maintain the universe, are those of numbers. The study of arithmetic is the path to perfection. Through it and the sect's precepts, the individual can discover new aspects of the divine plan and of the mathematical rules that govern the universe. From this principle comes the devotion to the study of numbers.

The fusion of religion and reasoning that began with this doctrine would give rise to the characteristic intellectualised theology of the West from Plato to St Augustine and Thomas Aquinas, and later with Descartes, Spinoza and Leibniz. From Pythagorean inspiration, Western philosophy and science had a moral aspiration for centuries and participated in a concept of the eternal world that could be revealed to the intellect and not just to the senses.

Mathematical harmony

Perhaps Pythagoras was not the first to envision the famous theory that bears his name, but he was certainly the first to give a decisive proof that showed it worked for any right-angled triangle. Although he owes his place in the history books to his mathematical discoveries, his achievements encompass a wide range of disciplines. For example, he is also said to have discovered the relationship between different lengths of string and musical tones.

This particular legend relates that walking down the street, Pythagoras heard the sound of hammering coming from a blacksmiths. He went and had a look and saw that the sound was coming from the vibrations of the metal being hit by the hammer; the longest pieces produced lower sounds. He experimented with bells and glasses of water and investigated the vibrations of strings and found a relationship: the pitch of a sound from the string is inversely proportional to its length.

If two pieces of string of an identical material and thickness are subjected to the same tension, but one is twice as long as the other, the shorter string will vibrate with twice the frequency as the longer one. In musical terms it is said that the notes produced by the strings are separated by an *octave*. The ratio between one string and the other an octave higher would be 2:1. Equally, if we reduce the difference between the lengths by half, we would have the musical interval called a *fifth*, which has a ratio of 3:2. The ratio of 4:3 corresponds to an interval of a *fourth*, etc. The terms *octave*, *fifth* and *fourth* come from the position of these intervals on the musical scale. This discovery is of utmost importance for music and science alike. It was the first natural phenomenon to be described in terms of a precise quantitative expression, in other words, the physical world converted into numbers.

If two strings vibrate simultaneously they produce a musical chord. From there it was only a small step to Pythagoras' next discovery. The master found that the combination of different string lengths – as in the *octave*, *fifth* and *fourth* – produced pleasant, or consonant, chords. The difference between the notes are described as 'perfect' intervals. Lengths with more complex proportions produce less pleasant chords. For example, the ratio 9:8 produces a second, which gives a dissonant chord. When Pythagoras concluded that numerical proportions governed the laws of musical harmony, he thought that the same must occur in the entire universe, which he himself named the *cosmos*, which means 'order'. The qualities of all things, be they material, musical, or abstract, such as justice, could be explained by numbers.

This may seem a very bold leap but maybe it is easier to understand if we take into account that in ancient Greece music was given the same importance as arithmetic, geometry and astronomy. From Plato to Aristotle, these four disciplines were considered the four pillars of wisdom and formed the medieval quadrivium ('four paths'), the basic curriculum for any cultured person, valid in European universities until the Renaissance. Since their union in the Pythagorean doctrine, mathematics and music have had a profound influence. For example the term *harmonic* is borrowed from music to describe a great number of mathematical phenomena – harmonic series, harmonic functions, harmonic motion etc.

The divine number

Chapter V of book I (Alpha) of Aristotle's *Metaphysics* is titled *The Pythagoreans and their Doctrine of Numbers*. The text is considered a primary source and experts consider it to be the most respected presentation of Pythagorean philosophy. The chapter begins with an introduction that is renowned for its clarity and precision:

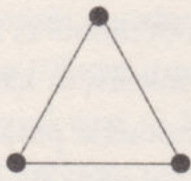
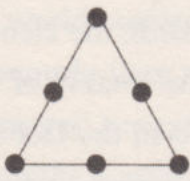
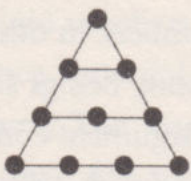
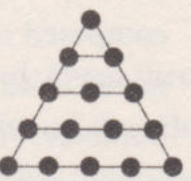
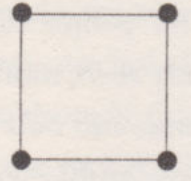
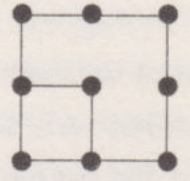
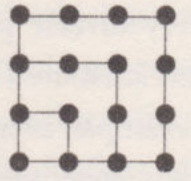
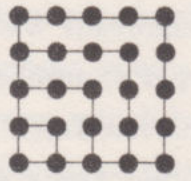
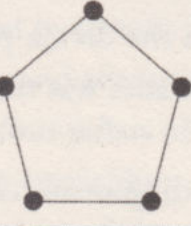
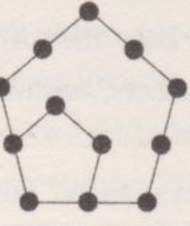
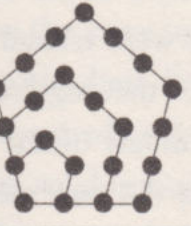
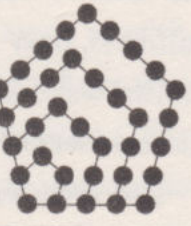
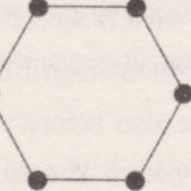
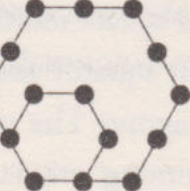
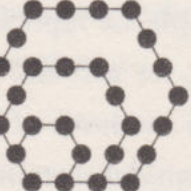
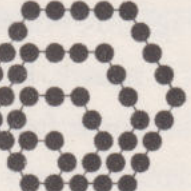
“The Pythagorean philosophers dedicated themselves to the study of mathematics and were the first to advance it ... They supposed that existing things are numbers – not numbers existing on their own, but that things are truly composed of numbers. In other words, the elements of numbers are the elements of all existing beings and the totality of the universe is harmony and numbers. Their argument consisted in the fact that numeric properties were inherent to the musical scale, to the heavens and many other things.”

Although it presents a monolithic impression, the numeric world of the Pythagoreans was a very diverse place. Pythagoras and his people identified many types of numbers and attributed them, as we have already seen, moral and physical characteristics. For example, even numbers were feminine, and odd numbers masculine. Some numbers were amicable and therefore compatible with each other, but others were evil and did not get on well with the rest – they could even bring bad luck to humanity. The job of arithmetic was to discover the different types of numbers, how they were related and how they fit into the divine plan. Pythagoreanism was a kind of theology of numbers, and mathematics was the theology that had uncovered the divine order.

But what exactly did they mean by *number*? The Pythagoreans had three definitions for it. A number is a “limited quantity”; a “combination or accumulation of units”; and an “amount that accrues”. Pythagoras imagined numbers as images or diagrams. For example, he represented square numbers with dots forming the shape of a square. In fact, we still talk about squares and cubes of numbers today, terms that come from the methods of Pythagoras. The master also references rectangular, triangular and pyramidal numbers, among others, in which the quantity of each number corresponds to the number of stones needed to make the shapes in question. It can be assumed that the master basically conceived the world as atomic, with bodies made from molecules, in turn formed by atoms, placed in solid shapes, in same way as he assembled numbers. Perhaps this is the reason why he even

considered beings to be “mobile numbers”. Beyond this mystical assumption, this type of geometric algebra was the precursor to today’s symbolic algebra.

The Pythagorean study of numbers began with a spiritual search, something like the Hebrew Kabbalah, where each number also has a symbolic significance. *One*, the monad, was the generator of all numbers, since every number can be created from it by repeated additions. It had a special status and was not considered to be a number itself. It was the monad, the symbol of reason, existence and stability. It was also associated with the right-hand side. *Two*, the dyad, meant diversity, the undefined. It

TRIANGLES					
	1	3	6	10	15
SQUARES					
	1	4	9	16	25
PENTAGONS					
	1	5	12	22	35
HEXAGONS					
	1	6	15	28	45

The Pythagoreans considered those numbers that could be used in the form of stones or beads to form polygons, such as the triangle, the square and the pentagon, to be part of the same family.

was associated with the left-hand side and was considered feminine. *Three*, the triad, was the union of the monad and the dyad, and therefore a symbol of harmony and perfection. It was associated with time and considered to be the masculine principal. *Four* meant universal and inexorable law, and therefore, justice, given that $4 = 2 + 2$. It was considered the key to nature and mankind.

Five was the union of the dyad and the triad, the feminine and the masculine, and thus, it was the symbol of marriage and of the divine triangle ($3^2 + 4^2 = 5^2$). Regular, solid polyhedrons with faces made of identical regu-

PYTHAGOREAN NUMBERS

The Pythagoreans meticulously classified numbers, sometimes around strange concepts, only understandable from the point of view of their mystical beliefs. Euclid brings them together and clarifies them in Book VII of *Elements*.

- Odd and even numbers They are divided into four classes:
 - Evenly even: those whose half is even.
 - Oddly even: those whose half is odd.
 - Evenly odd: those which, when divided by an odd number, give an even number.
 - Oddly odd: those which only have odd divisors
- Numbers that are divisors of others, or multiples of others.
- Prime numbers and composite numbers.
- Linear, plane, solid, squared and cubic numbers: *linear numbers* are those that have no divisors (actually, the prime numbers); *plane numbers* are the product of two numbers that form their sides; *solid numbers* are the product of three numbers that form their sides; *squared numbers* are the product of a number multiplied by itself; *cubic numbers* are the product of a number multiplied by itself twice.
- Perfect, deficient and abundant numbers: *deficient numbers* are those that are less than the sum of their aliquot parts (an aliquot part is that which is an exact divisor of a total, such as 2 with respect to 4); *abundant numbers* are those that are greater than the sum of their aliquot parts, and *perfect numbers* (as we have already seen) are those that are equal to the sum of their aliquot parts.
- Amicable numbers: those numbers where each number is equal to the sum of the divisors of the other. The Pythagoreans only knew 220 and 284.

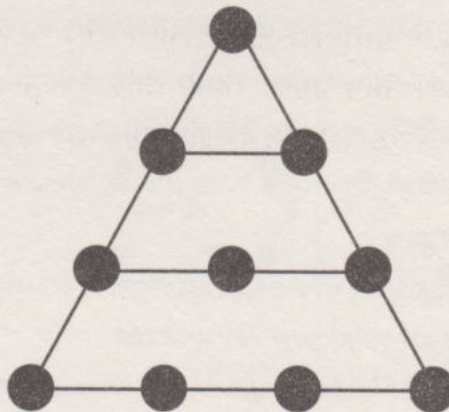
$$284 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 \text{ (the sum of the divisor of 220)}$$

$$220 = 1 + 2 + 4 + 71 + 142 \text{ (the sum of the divisors of 284)}$$

lar polygons were also five in number: the tetrahedron (4 triangles), the cube (6 squares), the octahedron (8 triangles), the dodechedron (12 pentagons) and the icosahedron (20 triangles). It represented the five elements said at the time to form the universe – fire, earth, air, water and the celestial sphere that surrounded everything. As we have seen, the number five had great importance. Therefore, in the form of a pentagram, the number became the Pythagorean's emblem, used by a members of the sect to recognise each other.

However, even more divine than five was *six*, the symbol of procreation and the product of the feminine and masculine ($6 = 2 \cdot 3$) and, above all, the first perfect number. A number is perfect if it is the sum of its divisors (including one but excluding the number itself). Therefore, the divisors of 6 are 1, 2 and 3. The Greeks were only aware of four perfect numbers. The other three are 28 ($= 1 + 2 + 4 + 7 + 14$), 496 ($= 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$) and 8,128 ($= 1 + 2 + 4 + + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1,016 + 2,032 + 4,064$). Currently, we know of 43 in total, all of them even. It is not known whether there are any odd perfect numbers or if the amount of perfect numbers is finite or infinite.

Seven symbolises virginity, since it is neither created nor able to create. Therefore, it is associated with health and light. There were seven roaming heavenly bodies (mostly planets) that gave the seven days of the week their name. *Eight* was the symbol of friendship, plenty and reflection. It was the first cubic number. *Nine* was the symbol of love and pregnancy. Finally, *ten*, the decade or tetractys, was the symbol of God and the universe. Given that, for Pythagoreans, the first four numbers contained the secret to the music scale, their sum ($1 + 2 + 3 + 4 = 10$) was considered perfection, the synthesis of “the whole nature of numbers”, in the words of Aristotle. As a beginning and basis for everything, the tetractys was the Pythagoreans' ultimate expression of



*The tetractys in the form of a triangle,
symbol of the Pythagorean oath.*

numeric mysticism, since, when represented in the shape of an equilateral triangle, the symbol was used when initiates were sworn into the sect.

The Pythagorean fixation with numbers had all kinds of consequences. Some were beneficial to the progress of knowledge. For example, the study of the mathematical properties of numbers (always positive integers in this case) sowed the seeds of modern number theory. But its fanatical obsession was evidently to its detriment. It blinded the Pythagoreans to ideas that were a better fit for explaining natural phenomena and subjected the laws of nature to the unrealisable ideals of beauty, symmetry and harmony. For centuries the immovable belief in the supremacy of numbers slowed progress that would have led to other, more suitable models to describe the complexities of the world.

The clearest result of this narrow-mindedness is the Greek theory of cosmology. The Pythagoreans believed that the Earth was a static globe suspended in the middle of the universe, while the stars – set like jewels in a celestial sphere – moved around in perfect circles every twenty-four hours. The Sun, the Moon and the five known planets spun around the Earth on their own circular orbits, within the celestial chamber.

By the 3rd century B.C. these reputed circular orbits did not match up with observations and data. They were then replaced by epicycles, small circles that moved around a central circular orbit. Over time, the number of epicycles grew until the system was so overloaded that it was ridiculous and, of course, completely useless. The possibility of the celestial bodies following any orbit other than a circular one was unacceptable for the Greeks. Orbits had to be in the form of a circle, the most perfect of all shapes.

Even when Copernicus, in his great work *De Revolutionibus Orbium Celestium* (On the Movement of the Celestial Orbs, 1543) dethroned the Earth from its position in the centre of the universe and replaced it with the Sun, he still kept the old circular orbits. It was not until 1609 that Johannes Kepler retired them, replacing them with ellipses, to which Isaac Newton would later add parabolas and hyperbolas.



The cosmological model of Ptolemy, who lived in the 2nd century A.D., followed the Greek model. This engraving from 1493 shows that the model was still valid in the 15th century.



Copernicus's solar system, presented in the seminal work *De Revolutionibus* still has not quite abandoned the circular orbits.

The legacy of the Pythagoreans

After the death of Pythagoras, the community continued to be active. His wife Theano and his daughters helped to maintain the traditions he had established. It is thought that Theano worked on the golden ratio, but her research has not been preserved. The school was divided into two groups, those who were occupied by mysticism and rituals, the acusmatics (from *akousmatikoi*, 'those who listen'), and those who were interested in pursuing the knowledge of numbers, the mathematicians (*mathematikoi*, 'those interested in knowledge').

The Pythagoreans were against the idea of an aristocracy but ended up constituting an aristocracy themselves, and they even intervened in political issues. This led to popular uprising against them in Croton. The Pythagoreans were persecuted and many of them killed. The communities split: the mathematicians went to Tarento, and the acusmatics became itinerant mystics. The dispersion of so much mathematical and philosophical talent would contribute to the formation of an unprecedented centre of intellectual activity, the first and most important of schools, the Platonic Academy. It may have been the end of the community, but the beginning of the immortality of Pythagoras.

Pythagoras' visions continued to influence not only Greek philosophy, but also all of Western thought. Plato's theories of the soul, his explanation for creation, the properties of things as pure ideas, are borrowed directly from Pythagoreanism.

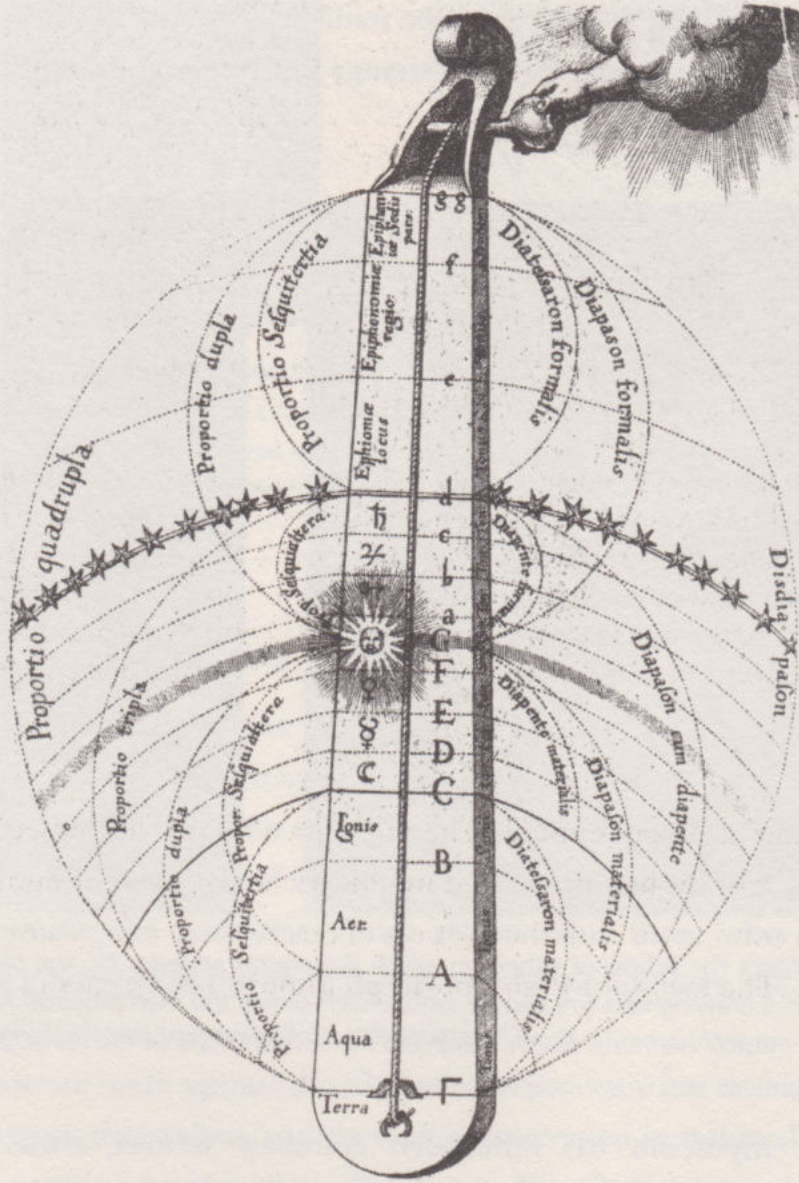
Pythagorean numerology also influenced Christian mysticism. Galileo was considered by his followers to be like Pythagoras. His famous quote “the book of nature is written in the language of mathematics” highlight his allegiance. Copernicus and Leibniz stated their faith in the same tradition. Newton devoted a large part of his life to alchemy and other similarly strange undertakings under the influence of Pythagorean mystic Jakob Böhme.



Details of Rafael's The School of Athens in which Pythagoras appears. On the slate held by one of his disciples a drawing of a lyre can be seen, which shows the mathematical fundamentals of musical harmony. Below it appears the sacred shape of the tetractys.

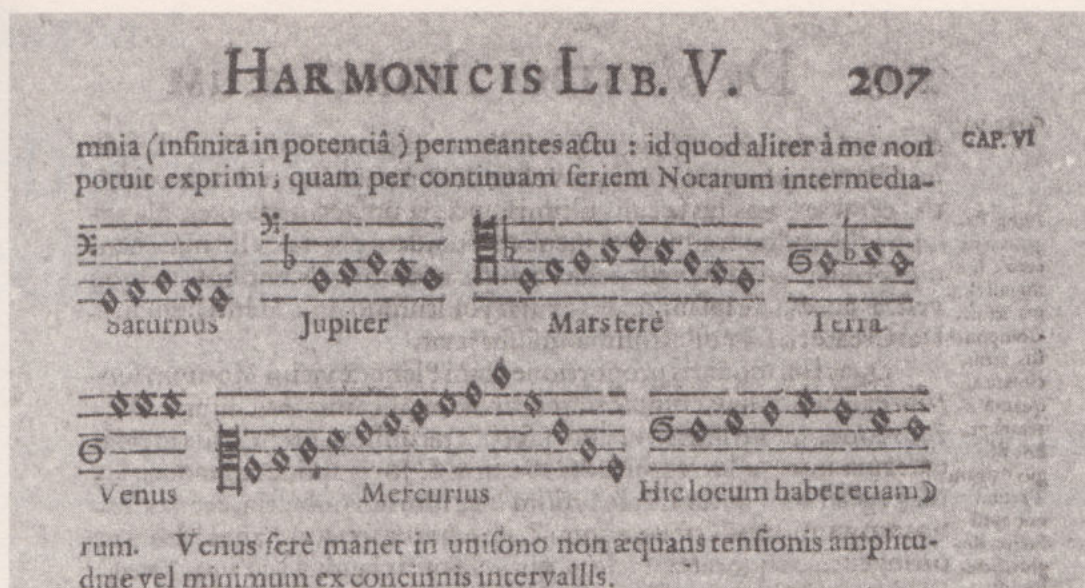
Numeric mysticism has influenced countless writers, artists and thinkers throughout the centuries, as also happened with the relationship between mathematics and music. In the Renaissance some cathedrals were designed following musical proportions 2:1, 3:2 and 4:3. A celebrated illustration of the work *Anatomiae Amphitheatrum*, published by the philosopher Robert Fludd in 1623, shows the hand of God tuning a celestial monochord. The divine hand is tightening the string on a board around which the planetary orbits overlap at intervals from the musical scale. The idea of the ‘music of spheres’ inspired many scientists, even the greatest and most revolutionary, including Johannes Kepler. A mystic to the core, Kepler spent (some would say wasted) thirty years of his life trying to discover the laws of planetary movement in musical harmony. He believed that every

planet played a different melody according to its distance from the Sun; the closer they were, the higher the notes.



The celestial monochord of Anatomiae Amphitheatrum, tuned by the hand of God. The work was published in 1623 by the English philosopher Robert Fludd.

Pythagoras' intellectual influence in ancient and also modern times make him one of the most important men ever to have lived, both when his thinking contained genuine wisdom and when it was utter rubbish. Mathematics as a deductive and demonstrative tool began with him, and within this it is combined with a strange form of mysticism. The notion that a mathematical equation corresponds to a real-life phenomenon is an expression of numeric Pythagoreanism. Thinking of recent scientists who have made contributions of indisputable importance, we



In search of a fundamental principal that would explain the reason for the orbital eccentricities of the planets, Kepler measured all of their the maximum velocities at the perihelion (the point closest to the Sun) and then minimum velocity at the aphelion (the furthest point). To the astronomer's delight, the ratio between one velocity and the other corresponded at harmonic intervals. In this image of a Latin edition of Harmonia Mundi, Kepler represented the ratios in the form of musical notation, a definite homage to the Pythagorean 'harmony of the spheres'.

obviously think of Albert Einstein. The physicist who revolutionised the 20th century spent most of his life intoxicated by this particular form of numeric Pythagoreanism until coming up with the formula of relativity $E = mc^2$ (energy = mass $\times$ the speed of light²). The legacy of Pythagorean philosophy still dazzles us today.

Chapter 2

History's most celebrated theorem

Everything is a number
Pythagorean principle

The first question that arises from any mathematical problem is: What is its use? Often the answer can be found by answering a more basic question: Where does the problem arise? The more abstract the question, the greater the need to find more fundamental answers in order to understand it and make progress.

One example is the roots of numbers. Coming up against these for the first time provokes a brief moment of confusion. Attempting to find the best way to solve them, searching for the initial solutions, the question arises: What can these roots be used for? However, the question should really be: Where do the roots of numbers come from?

As always, the answer is simple. The roots of numbers appear in extremely simple geometric problems, such as the calculation of the diagonals of squares and rectangles based on the length of their sides. Pythagoras' theorem is a famous answer to a classic geometric problem. Part of its importance also stems from the fact that the theorem is vital for the introduction and development of square roots. This is why it is the gateway to the vast mathematical territory populated by roots of numbers.

Good morning, numbers!

Man created numbers to solve his problems of counting and measurement. First came the natural numbers (1, 2, 3...), to count and order. Then fractions arose as a result of comparing these quantities: $1/2$, $3/4$, $5/8$... And so began the task of describing every point of a straight line with numbers, a task that is still ongoing in mathematics today.

THE SYMBOL $\sqrt{}$

This curious symbol that is now so familiar to us was the invention of Christoff Rudolff, published in 1525. It is believed that the shape was based on the letter r (an abbreviation for radical). Note that in general $\sqrt{x}$ has two values $\pm\sqrt{x}$ although when dealing with positive numbers, $\sqrt{x}$ is understood to refer to $+\sqrt{x}$.

With the birth of numbers, man entered an abstract world, one filled with equations, through which he was able to explain and order reality (or at least try to). However, his journey through this world in search of certainties was crammed with problems and doubts from the outset. In the beginning, for example, man had a dream: everything was measurable, if not using natural numbers, then using fractions. However, as the science of numbers advanced, it became evident that a premise such as this was not necessarily possible. It was in geometry, in measuring a variety of lengths, where the problem quickly appeared. Searching for something as simple as the diagonal of a square whose sides were only 1 unit long, it immediately became clear that this magnitude could not be expressed as a fraction or a natural number. Suddenly, it became evident that there might be vast numbers of lengths for which it was not possible to measure using repetitions of the same unit, however small this might be.

The diagonal of a square with sides 1 unit long is $\sqrt{2}$, a number that caused a storm among the mathematicians of the time. It cannot be described as "natural" and, with time, has come to be called 'irrational', an unfortunate expression that nonetheless allows it to be distinguished from rational numbers (those which can be expressed using simple fractions). Following its discovery, other irrational numbers began to crowd the frontiers of mathematics. The progressive discovery of this type of number, the irrationals, remains one of the great mathematical challenges. Just as with $\sqrt{2}$, some of the best known irrational numbers appear in simple shapes, that can be drawn with a ruler and a compass. This is the case for the golden ratio $(1+\sqrt{5})/2$. Over the centuries, the number pi (π) has given rise to a special numerical devotion. Pi is the mysterious ratio between the perimeter of a circle (circumference) and the distance across it (the diameter). Today, calculation of its decimal figures runs into the billions and is a challenge being tackled by supercomputers.

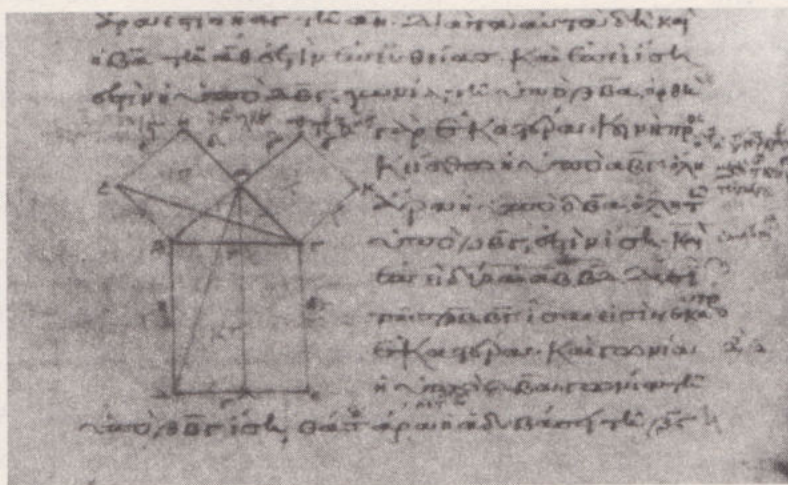
Square and cubic roots first arose in geometric problems, although they soon cropped up elsewhere. They appeared, for example, together with a new mathematical discovery that involved yet another problem to be solved: third and second degree

equations. When these equations appeared, they caught mathematicians off guard to such an extent that no one knew a method to solve them, and these had to be specially created. These methods formed the basis of a new science: algebra. Number roots contributed to the birth of a new mathematical discipline the influence of which has since extended into every classroom, science lab and factory.

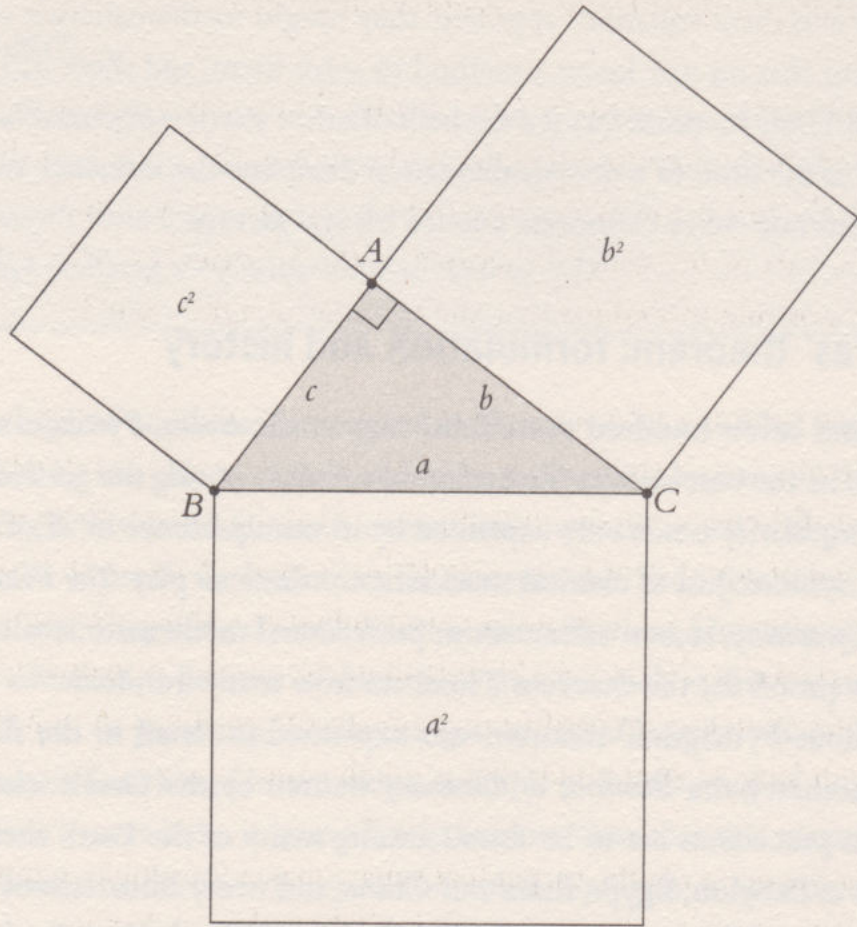
Pythagoras' theorem: formulation and history

Two thousand seven hundred years following its discovery, Pythagoras' theorem continues to be the best known mathematical solution among the general public. Its enormous popularity is not only explained by its omnipresence in all of the world's educational systems. Just as classical musicians continue to play *The Four Seasons* or the *Ninth Symphony*, season after season, professional mathematicians continue to publish new proofs for the theorem. There are now many hundred.

The famous Pythagoras' theorem was explained in detail in the first Western book on geometry, the *Elements of Geometry* written by the Greek scholar Euclid. However, its precedents are to be found among many of the East's ancient civilisations, such as Babylon, Egypt, India and China, and many historians of science are convinced that Pythagoras must have been aware of these ideas from his travels. In spite of all this, the genius of the Greek mathematician cannot be denied. The preceding knowledge was based on specific examples, mathematical results from geometric figures. It was Pythagoras who made the great leap from the specific to the general. He went from specific examples of triangles to a general theorem that held for any right-angled triangle. Like all the Greek geometers, he established a theoretical schema that could be applied to all cases.



Section of manuscript for Euclid's *Elements of Geometry* which describes Pythagoras' Theorem.



In its most authentic form, Pythagoras' theorem can be stated as: Given a triangle with vertices (corners) ABC , the angle A is a right angle if and only if the area of the square of the side a , opposite A , is the sum of the areas of the squares of the other two sides b and c ($a^2 = b^2 + c^2$).

CATHETI AND HYPOTENUSE?

These words describe the sides of a right-angled triangle and, etymologically speaking, have their roots in the Greek language. Cathetus comes from *kathetos*, which is derived from *kothos*, meaning straight, in agreement or perpendicular. Hypotenuse comes from the Greek word *upoteinousa*, which means 'the line that underlies or supports'. This definition supposes that the hypotenuse is the diameter of a circumference in which the right-angle vertex of the right-angled triangle is to be found, or rather, the diameter underlies the right angle.

Based on this, it is possible to make a few initial remarks:

- a) The following statement is often made: "For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the catheti." Effectively this happens because the catheti are perpendicular. But this is just one part of the general theorem. If the equation $(a^2 = b^2 + c^2)$ holds, it is then possible to deduce that the triangle is right angled.
- b) First and foremost, the theorem serves to determine perpendicularity. Imagine that a Greek architect decides to check if two walls are perpendicular. First of all, he picks up the instrument used at the time to measure lengths: a cord tied in equidistant knots. Using this cord, he marks 3 units on one wall and 4 on the other. The walls are perpendicular if there are 5 units between the marked ends $(5^2 = 3^2 + 4^2)$. It is an ingenious way of reducing a problem of angles, which were hard to measure, to the verification of a relationship between lengths, something that is much easier to do.

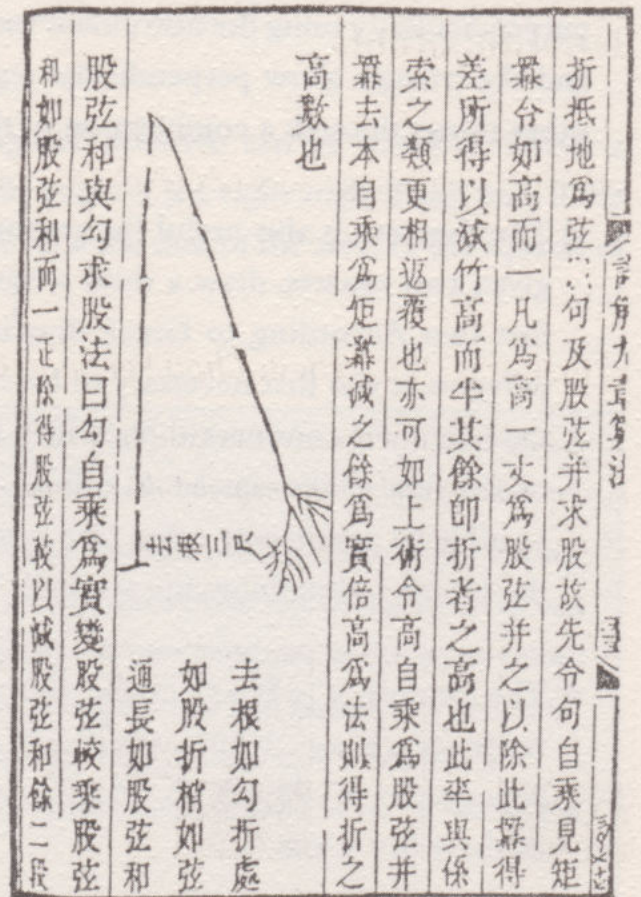
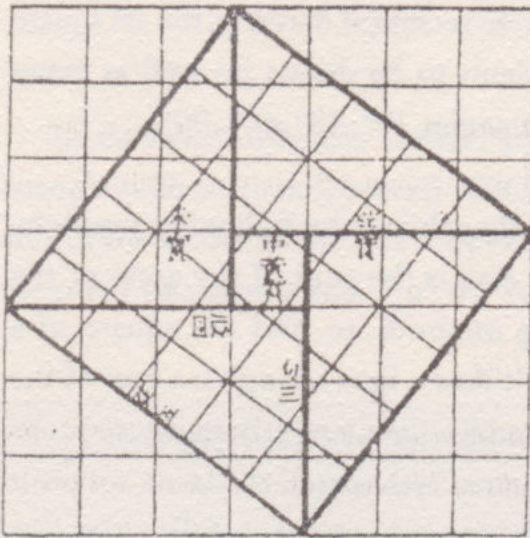
Note that the design of the carpenter's square makes it possible to directly verify perpendicularity using the instrument itself. In technical drawing, the set square and the triangle allow perpendicular segments to be drawn (as well as many other things through a combination of the angles 30° , 45° , 60° , 90°).

- c) The theorem is also useful for graphically solving the following problem: given two squares, draw a third whose area is the sum of the areas of the first two. According to Greek drawing methods, to find the square of a polygon, it was first necessary to break it down into triangles; a box of the same area was constructed from each triangle; then everything was reduced to the sum of the squares in a final square. Pythagoras' theorem makes it possible to graphically solve "the sum of squares" without difficulties (see the proofs in the following section).
- d) The formula $a^2 = b^2 + c^2$ implies the requirement for the extraction of square roots: $a = \sqrt{b^2 + c^2}$ while $b = \sqrt{a^2 - c^2}$ and $c = \sqrt{a^2 - b^2}$. And in particular, the appearances of $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7} \dots$ which necessarily lead to the discovery of these first roots.

As we have indicated, Pythagoras (or a modest follower who discovered the result and attributed it to the dear leader) was commendable enough to note that the validity of this relationship for all possible right-angled triangles. However, specific versions of the theorem ($5^2 = 3^2 + 4^2$) were known in Babylon and Egypt many years before. For their part, the age-old Chinese and Hindu cultures were also quick to discover this geometrical property. In reality, the problem of the diagonals of squares has appeared independently in several cultures.

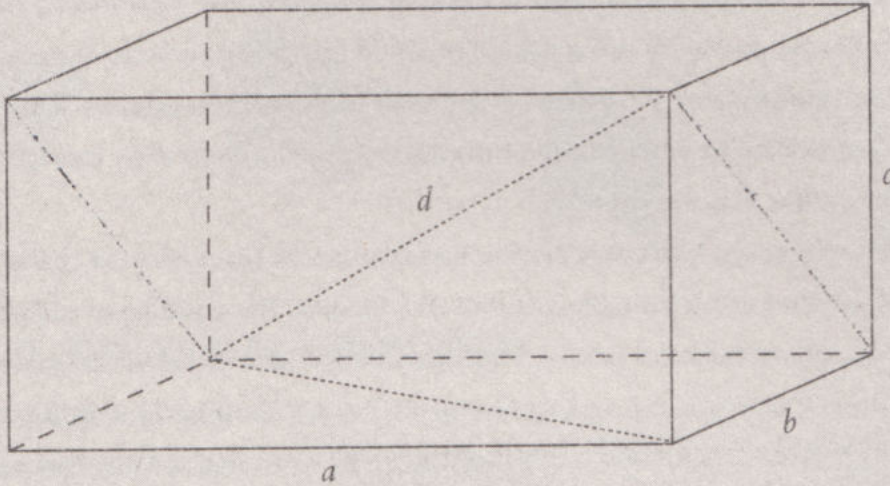
In China, Pythagoras' theorem is known as *Kon Ku*. The Chinese mathematical treatise entitled *Chou Pei Suan Ching*, which dates back to the 3rd century B.C., contains the drawing seen below left. This shows the equality of the area of the square with length 5 with the total of the squares with lengths 3 and 4 which, together with this side, form the 5 units of the right-angled triangle ($5^2 = 3^2 + 4^2$).

Years later, the mathematicians Chao Ching Ching and Liu Hiu proved the result for any right-angled triangle. They did so figuratively, making use of the image of a bamboo cane bending in the wind. If the vertical height of the bamboo stick and the horizontal distance between its base and branched tip were known, it was then possible to calculate the length of the cane (hypotenuse).



PYTHAGORAS' THEOREM IN SPACE

Pythagoras' theorem is normally associated with planar right-angled triangles. What would be a reasonable version of the theorem in three dimensional space? This problem has a number of answers. The most popular is a box with edges a , b , c with the value d of the diagonal of the box being expressed using Pythagoras' theorem: $d^2 = a^2 + b^2 + c^2$. Other possible versions will be considered in the final chapter.

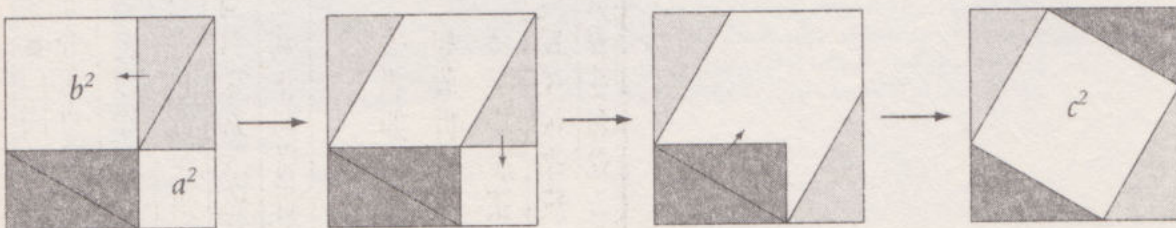


Beautiful proofs

The classical demonstrations of the Pythagoras' theorem are not only of mathematical interest. Both the ingenuity of the solutions and the elegance of their demonstrations has a satisfying beauty to it.

Pythagoras' theorem in *Chou Pei Suan Ching*

This Chinese document, the title of which can be roughly translated as *The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*, offers a beautiful proof using only the movement of shapes.



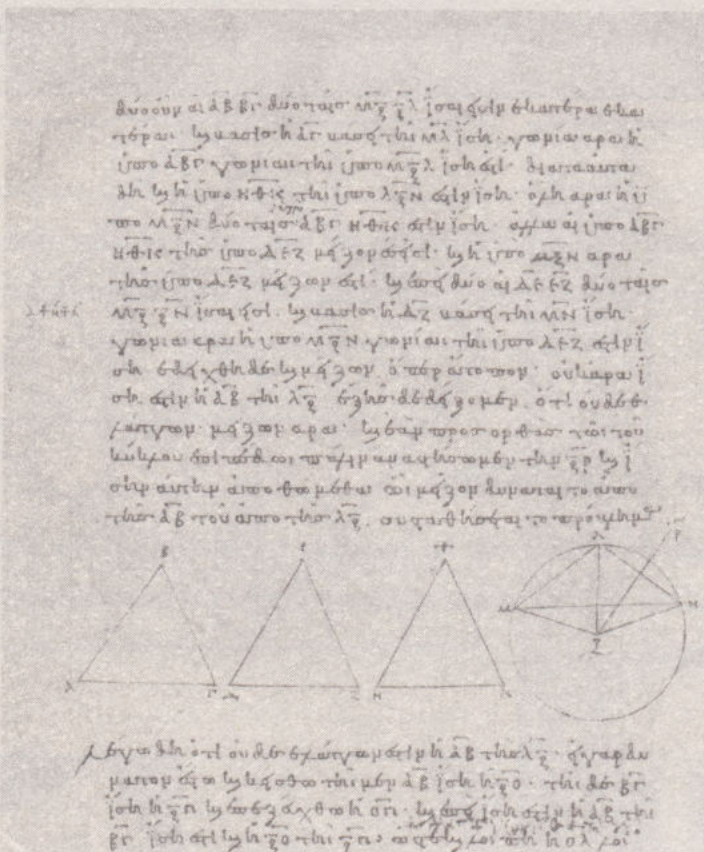
EUCLID, PYTHAGORAS AND THE *ELEMENTS OF GEOMETRY*

Euclid lived in Alexandria around 300 B.C. He was author of a work that is fundamental to the development of mathematics and science, the *Elements of Geometry*. In the book, he presents all the geometric knowledge of the time with elegant rigour, logically deducing all the main properties (theorems) from definitions, postulates and axioms. Euclid's work is not only a brilliant synthesis but the rubric for a large task of structuring geometric thought. This is why, to this day, in spite of the intervening centuries, it remains the fundamental reference point when learning geometry. Surpassed only by the Bible, the *Elements of Geometry* is the work with the most editions and versions, both in terms of hand-written copies predating printing and with more than one thousand editions in printed form.

The work is structured into volumes. The first four books are dedicated to basic plane geometry, which includes the congruence of triangles, the equality of areas and, of course, Pythagoras' theorem (Book I, Proposition 47). It also includes the golden ratio,

the circle, regular polygons and some quadratures (construction of squares).

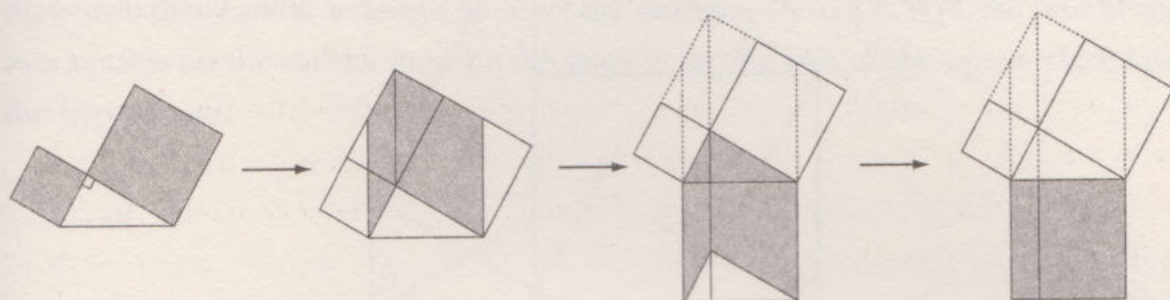
As such, the Pythagorean theorem is presented in a geometric context in a section about the area of shapes. Pythagoras reappears in Book VI in relation to ratios and also in Book X, which deals with square roots.



A page from Euclid's *Elements of Geometry* belonging to the so-called d'Orville Manuscript, written in Greek in Constantinople in the 9th century.

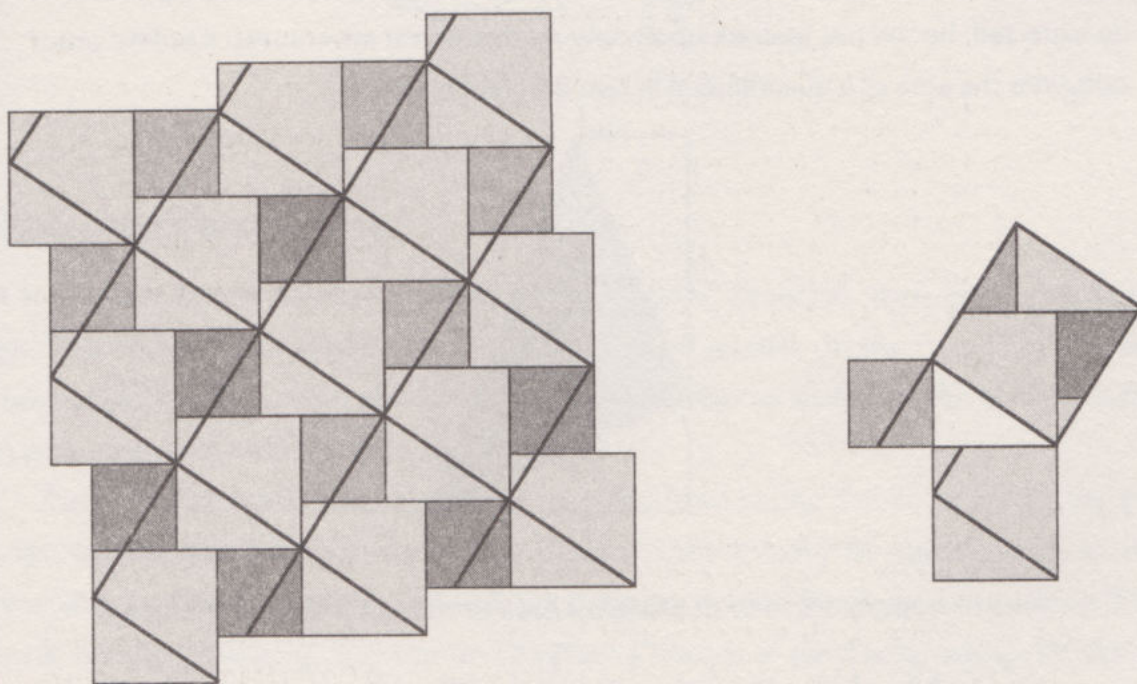
Pythagoras' theorem through the eyes of Euclid

Euclid presents the following graphical approximation of the problem. The Greek transforms each square on the catheti into a parallelogram with the same area (given that they have the same base and height). He then moves each of these parallelograms to the square on the hypotenuse. This clever proof clarifies the part occupied by each of the squares on the catheti.



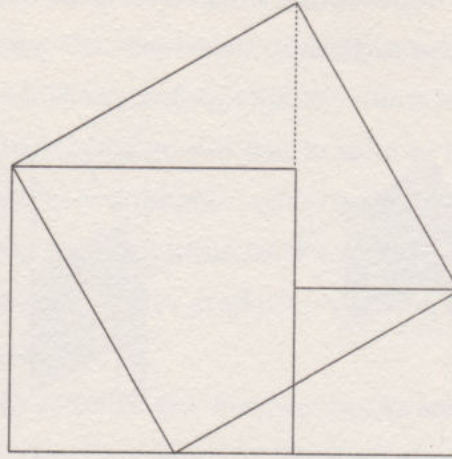
Pythagoras' theorem in an Arabian mosaic

This beautiful proof based on mosaics is attributed to Annairizi of Arabia (900 B.C.). The mosaic in the background is made up of the squares of the catheti, and the superimposed one, those of the hypotenuse. Both cover the same surface – a completely original resource to prove the theorem.



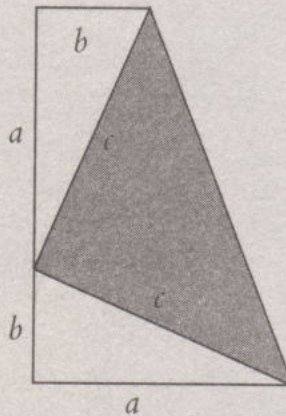
Pythagoras' theorem as seen by Henry Perigal

Another of the jewels in the collection is the proof of the English mathematician, Henry Perigal from 1873. The marvel is found in the creation of a puzzle that divides the large square into two sections.



A PROOF FROM THE PRESIDENT OF THE UNITED STATES

In 1876, before becoming the 20th president of the United States, James Abram Garfield (1831-1881) discovered and published an original proof of Pythagoras' theorem, which he debated with a number of colleagues in Congress. During his presidency, as was to be expected, he did not embark upon new mathematical adventures. Abrams' proof calculates the area of a quadrilateral in two different ways:



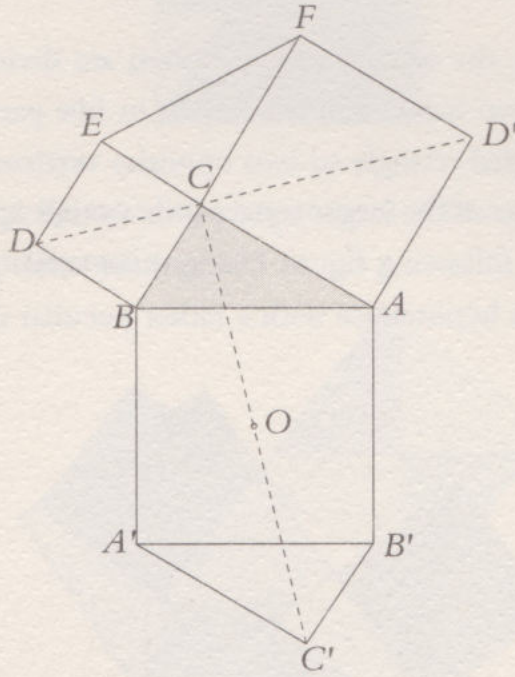
Area of trapezoid = Area of triangle₁ + Area of triangle₂ + Area of triangle₃

$$\frac{1}{2}(a+b)(a+b) = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}cc, \quad \frac{1}{2}(a^2 + 2ab + b^2) = \frac{1}{2}(ab + ab + c^2)$$

which gives $a^2 + b^2 + 2ab = 2ab + c^2$ or rather, $a^2 + b^2 = c^2$.

Pythagoras' theorem as proved by Leonardo da Vinci

The polymath Leonardo da Vinci (1452-1519) provided a brilliant proof of the theorem. He drew the triangle and its three squares on the sides, adding the part ECF above and another strategically placed copy $A'C'B'$ of the triangle below. Tracing DD' and CC' , which are perpendicular, it can be observed that DD' symmetrically divides the upper hexagon $ABDEFD'A$. If the two parts are rotated and flipped so they can be arranged to cover the hexagon $ACBA'C'B'A$, the sum of the two squares on the catheti must be the same as for the area of the square placed on the hypotenuse.



Other proofs and puzzles

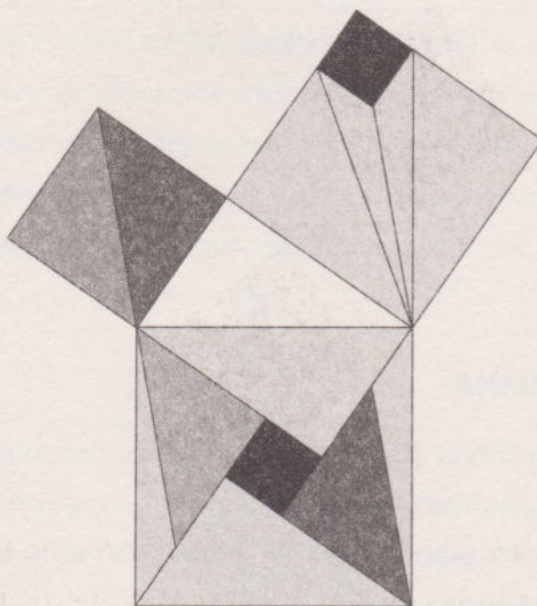
It is possible to obtain direct proofs of Pythagoras' theorem with great simplicity and beauty. All that is required is to divide the two squares on the two catheti of the right-angled triangle into pieces and try to use them to make up the square of the hypotenuse, as if completing a jigsaw puzzle.

The Chinese mathematician, Liu Hui, who lived in the 3rd century, during the reign of Wei, published an important book on the history of mathematics in the year 263, in which he presented various solutions to the problems of the time. The book is called *Jiuzhang Suanshu* or *The Nine Chapters of the Mathematical Art*. As an example, it includes an approximation of pi as 3.141014, and the author suggests the number 3.14 as the best representation of this constant. There is also a proof of Pythagoras' theorem in the form of a puzzle.

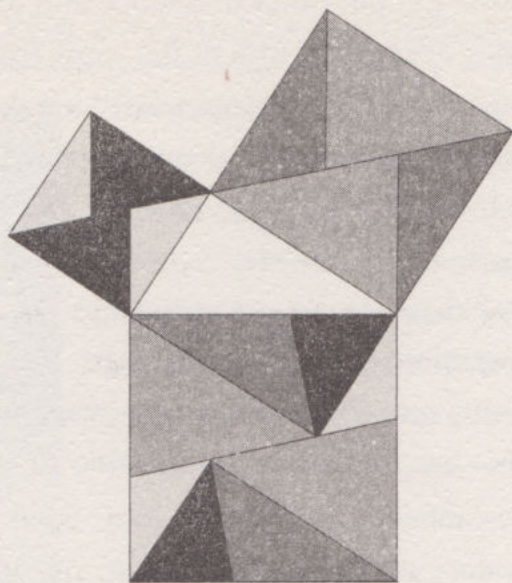
PYTHAGOREAN PUZZLES

It is possible to play a game with the puzzles that prove the theorem. The game makes use of a square photograph the size of the square of the hypotenuse. The photograph is arranged on the square of the hypotenuse and cut up along the lines in the proof figure and repositions in the squares of the catheti. The player is then required to restore the photograph in the larger square. Upon completion, the content of the photograph is revealed and the most famous theory in history has also been proven.

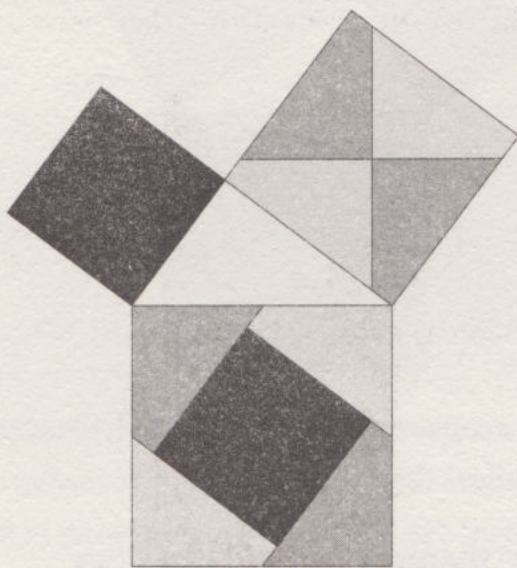
In this curious puzzle, the squares of the catheti are divided in the following manner: the smallest, in two halves and the largest in five parts obtained by moving the original right-angled triangle to two opposite vertices of the larger square and dividing the remainder of the larger square into a small square and two further triangles, as shown in the following figure. Using these seven parts, it is possible to make up the square of the hypotenuse with a rather peculiar distribution.



Johannes Eduard Böttcher (1847-1919) was a German physicist who became rector of the Leipzig Realgymnasium, despite dedicating a large part of his research time to pure mathematics. In 1886 he published a small article in a magazine about mathematics and natural sciences entitled "A simple model for the demonstration of the theorem of Pythagoras". The article proposed a puzzle that divided each of the squares of the catheti into four triangles, two and two, and with these it is



possible to obtain the square of the hypotenuse with a curious central symmetry, as shown in the next illustration.

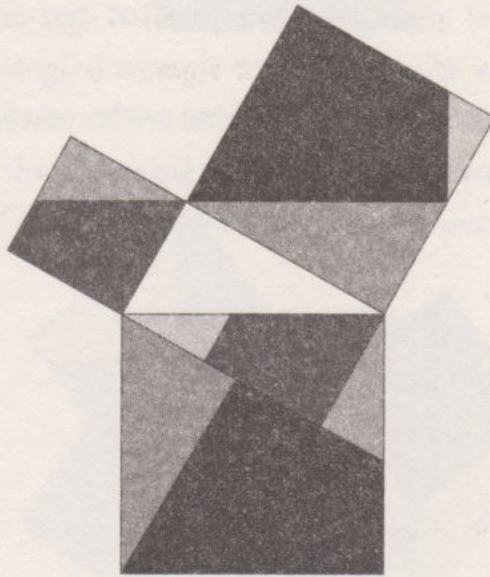


In recreational mathematics, the great British inventor, Henry Ernest Dudeney (1857-1930) stands out in his own right, despite, somewhat surprisingly, lacking a mathematical education; he was completely self-taught. Dudeney invented and published a large number of puzzles and other ingenious games in the newspapers of the time, which were then collected in sizeable volumes now considered canonical texts for lovers of mathematical challenges. Pythagoras is to be found in many of them.

The following five-part puzzle makes it possible to assemble the square on the hypotenuse with the small square and the dissection of the other into four parts from its centre.

The American professor and publisher of science and philosophy, Elisha Scott Loomis (1852–1940) is not necessarily one of the most famous authors in mathematical circles. There is no equation or theorem with her name and her numerous publications have been forgotten, except for one notable exception: *Pythagorean Proposition*, published in 1927, which brings together and classifies 371 proofs of Pythagoras' theorem.

Tracing the line parallel to the hypotenuse from the right-angle vertex of the triangle divides the smaller square into two parts. It is possible to divide the square of the other cathetus into three with a line perpendicular to the first. Arranging the five parts that are obtained, it is possible to make up the square of the hypotenuse.



As can be seen, there are many diverse proofs of Pythagoras' theorem, and indeed, there are people who collect them.

A note on Pythagoras' theorem and parallel lines

Among the many postulates chosen by Euclid to develop geometry using deductive logic is the famous and much discussed fifth postulate:

"Only one line parallel to a given line can be traced through a single point."

In the previous pages, we have presented an overview of the proofs of Pythagoras' theorem. If we are to reflect upon these, we will be able to observe that all rely on certain properties. For example, that parallelograms with the same base and height have the same area, the sides of similar triangles are proportional, or

LEWIS CARROLL AND A PYTHAGOREAN PROBLEM



Lewis Carroll (1832-1898), the famous author of *Alice in Wonderland*, was an Englishman whose real name was Charles Lutwidge Dodgson and who dedicated his life to logic, mathematics, photography and, of course, to writing all sorts of fantastic tales and stories.

In 1850, he devised and published the following geometrical problem: In a 21 x 21 field, with four points A, B, C, D, from which points is the sum of the distances of the other three at a minimum?

In order to resolve the problem, Carroll made use of Pythagoras' theorem, making the following calculations:

$$AB = \sqrt{12^2 + 5^2} = \sqrt{169} = 13;$$

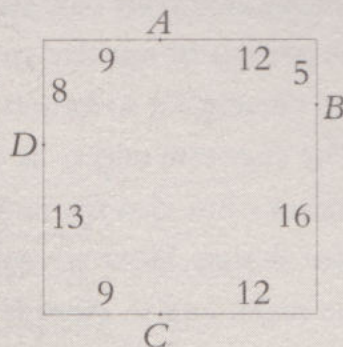
$$AC = 21$$

$$AD = \sqrt{9^2 + 8^2} = \sqrt{145} = 12+;$$

$$BC = \sqrt{16^2 + 12^2} = \sqrt{400} = 20;$$

$$BD = \sqrt{3^2 + 21^2} = \sqrt{450} = 21+;$$

$$CD = \sqrt{9^2 + 13^2} = \sqrt{250} = 15+;$$



With 12+, 21+ and 15+, Carroll wishes to indicate that these three roots give slightly more than 12, 21 or 15.

He then adds: "Thus the sum of the distances from A to the other points is between 46 and 47; from B it is between 54 and 55; from C it is between 56 and 57; and from D it is between 48 and 51 (Why not between 48 and 49? Think of this on your own). As such, the sum is smallest from A."

that the acute angles of a right-angled triangle are complementary. This allows us to argue that all the proofs of the theorem refer directly or indirectly to the fifth parallel postulate. The surprise is that (while taking Euclid's other postulates into account), Pythagoras' theorem itself is equivalent to the fifth postulate.

There have been many mathematicians throughout history who have studied whether it is really possible to deduce this postulate from the others, whether it still needs to be included, or whether other versions are possible. When this

statement is confined to planar space, where lines are straight, the fifth postulate is highly intuitive. However, if we consider other non-planar spaces, such as, for example, the surface of a sphere where the lines are always following a circumference, there is no line that is parallel to the one given.

Based on these considerations, replacing the fifth postulate with other concepts, such as there being no parallel lines or perhaps only a few, gives birth to so-called non-Euclidean geometries. As well as their theoretical interest, these other geometries can be used to model surfaces that are not flat planes with straight lines as we would understand them, but curved faces of a sphere, ellipsoid or parabola.

Current usage of Pythagoras' theorem

Even though some 2,500 years have passed since its discovery, Pythagoras' theorem continues to be present in our day-to-day lives. The only reasonable explanation for remaining so current for so long is that it is a fundamental solution with many applications. The theorem serves as an aid not only in solving problems from mathematical theory but also for making aesthetic and artistic decisions in the world of interior design. Next we will consider some of these applications, both in the academic realm and in everyday life.

Mathematical and scientific applications

Pythagoras' theorem is essential in the calculation of the lengths, areas and volumes of shapes. For a square with length x , the diagonal is $x\sqrt{2}$; for a rectangle with sides x, y the diagonal is $\sqrt{x^2 + y^2}$; for a parallelepiped (imagine a shoe box) with edges x, y, z the diagonal is $\sqrt{x^2 + y^2 + z^2}$; for a cone with height h with a base of radius r , the generatrix is $\sqrt{h^2 + r^2}$... The full list could fill the rest of the book by itself.

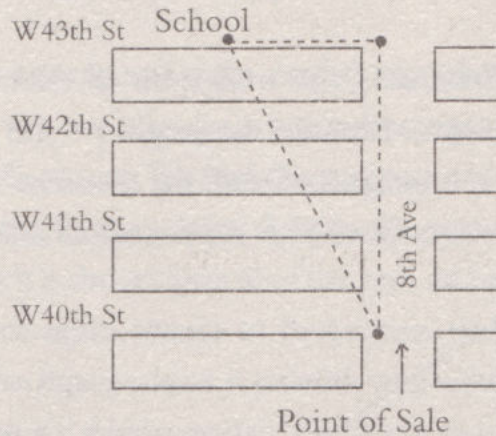
We also find Pythagoras present when we look at Cartesian coordinates on the plane (and in space) to define the distance $d(P, Q)$ between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$.

PYTHAGORAS ON THE STREETS

As human beings cannot currently easily pass through walls, in real life it is necessary to distinguish the points where it is possible to pass along the associated right-angled segment from those where it is not. Whatever the route the actual distance being covered is calculated as the shortest line joining both points.

This problem has considerable effects in big cities. On the one hand, it affects traffic in terms of the itineraries for the distribution of post and collection of rubbish etc. However it also determines town planning guidelines used to establish the distance between chemists, or between sex shops and schools, the location of health centres or other public services, etc. The problems of distances in the city has sometimes become a very real issue in court.

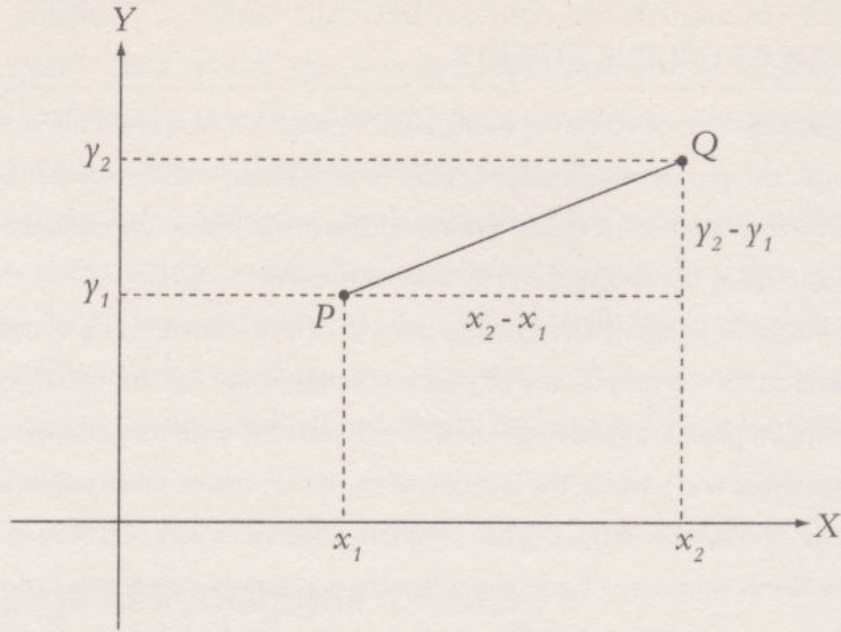
In 2002, the North American citizen James Roblins was arrested on a street corner in Manhattan, 8th Avenue and West 40th Street. He was accused of selling drugs with the aggravating circumstance of being less than 1,000 steps from the Holy Cross school, located on 43rd Street between 8th and 9th Avenue.



To justify this unacceptable proximity to the school, the police used Pythagoras' theorem $a^2 + b^2 = c^2$ using $a = 764$ steps (distance from the point of sale on 8th Avenue), $b = 490$ steps (distance to the school along 43rd Street) as the catheti, making the hypotenuse $c = 907.63$ steps. And therein lies the crime. That's less than 1,000 steps!

The defence lawyers argued that the "distance of less than 1,000 steps" had to be measured on a path that could be walked in real life, without passing through walls. As such, the aggravating factor did not exist ($a + b = 764 \text{ steps} + 490 \text{ steps} = 1,254 \text{ steps}$).

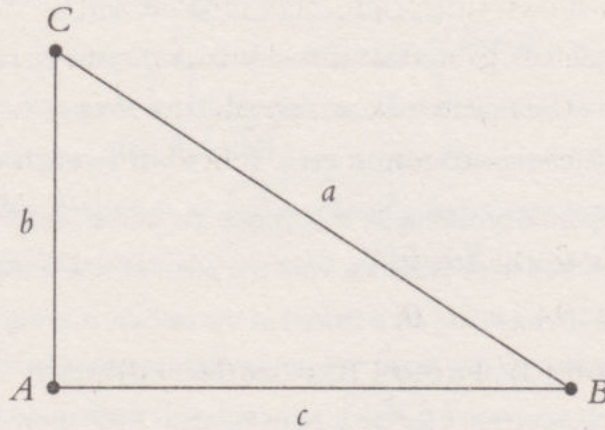
The Courts of Appeal ruled in favour of the police, meaning that the New York justice system is now Pythagorean.



Applying the theorem:

$$\text{Distance } (P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

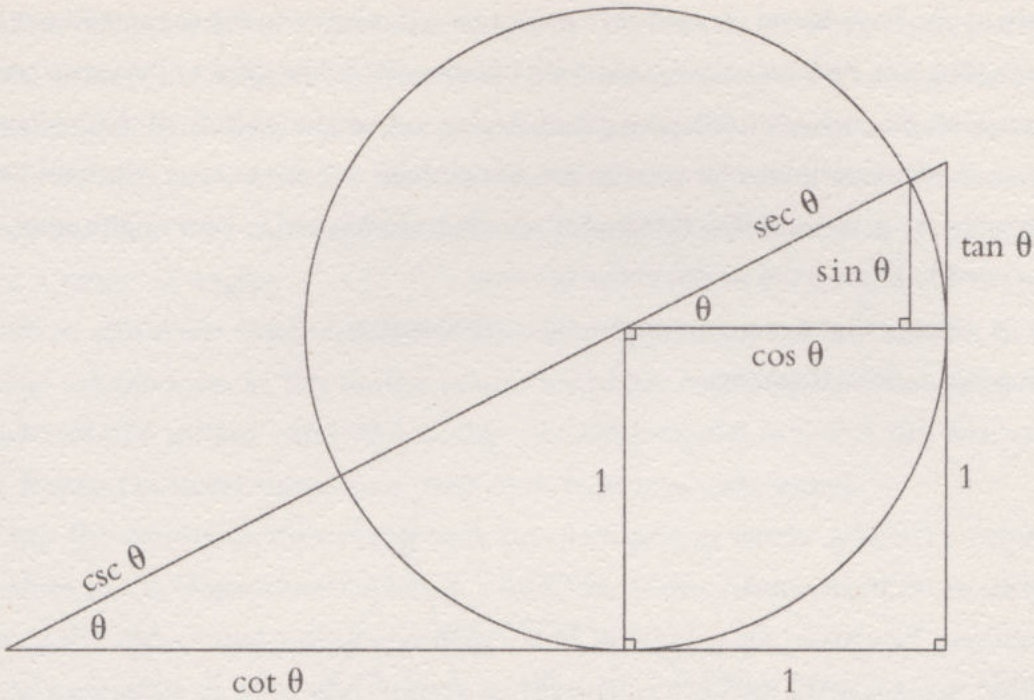
The Pythagorean relationship also underpins all calculations that make use of functions (functional calculus). Consider the graphs $y = f(x)$ in Cartesian coordinates. Similarly, trigonometry is impregnated with the theorem. Insofar as the angles of a right-angled triangle are associated with the functions sine, cosine, tangent, etc



being $\sin B = \frac{b}{a}$, $\cos B = \frac{c}{a}$ and therefore the theorem of Pythagoras, in trigonometric terms is the famous $\sin^2 B + \cos^2 B = 1$ relationship.

As a result, Pythagoras is also present in topography (relief), cartography (production of maps), air or sea navigation problems, and of course in architecture, engineering and any type of activity that requires measurements.

Let us now consider the following figure:



In it we not only see a circle and right-angled triangle the catheti of which are the sine and cosine, but we can also see many other segments that correspond to the other trigonometric functions. We find the tangent, which is the quotient of the sine and cosine; but also the three reciprocal ratios, the secant, which is one over the cosine, the cosecant, which is the reciprocal of the sine, and the cotangent, the reciprocal of the tangent. Once again, thanks to Pythagoras, the variety of right-angled triangles thrown up by the figure makes it possible to produce a series of interesting equations that relate these six trigonometric functions to each other.

$$\tan^2 \theta + 1 = \sec^2 \theta,$$

$$\cot^2 \theta + 1 = \csc^2 \theta,$$

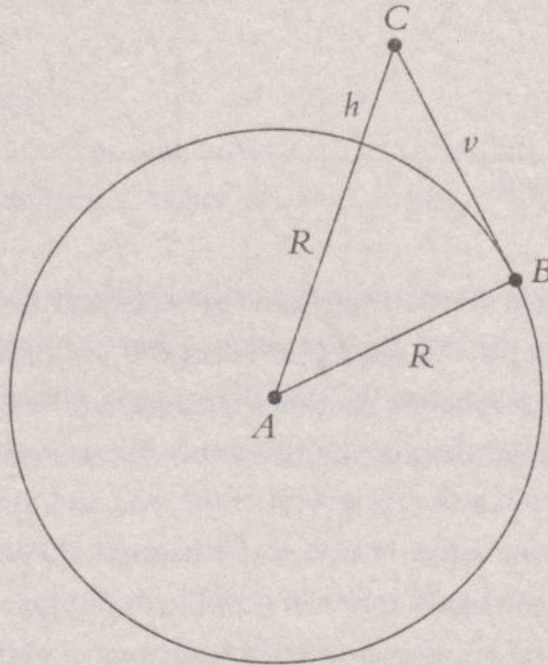
$$(\tan \theta + 1)^2 + (\cot \theta + 1)^2 = (\sec \theta + \csc \theta)^2.$$

WHERE IS THE HORIZON?

Perhaps you have found yourself on a mountain top, observing the sea spread out beneath your feet. Perhaps below you will see beaches with white sand, or the waves crashing against the rocks. In front, your gaze becomes lost on the horizon. At that moment, you may ask yourself how far you can see. But perhaps not; the scenery may have been too captivating. Nevertheless Pythagoras' theorem can be used to clear up the point. All you need to know is the height above sea level.

Let us suppose that the mountain measures 1,000 metres.

Using Pythagoras' theorem:



$$(R+h)^2 = R^2 + v^2.$$

That becomes $v^2 = (R+h)^2 - R^2 = (R^2 + 2Rh + h^2) - R^2 = h^2 + 2Rh = h(h + 2R)$, where R is the radius of the Earth.

As $2R + h$ is approximately $2R$ because h is extremely small with respect to R , thus:

$$v^2 \approx h \cdot (2R)$$

$$v \approx \sqrt{2Rh}$$

And with $R = 6,371$ kilometres and $h = 1$ kilometre so $v = 112.88$ km.

The origins of trigonometry date back to Babylon and Egypt. The discipline arose from the calculations used to make sense of the unexpected astronomical observations made by these civilisations. The methods were then put to use solving practical problems in the field of architecture and surveying, measuring shapes, heights and distances accurately. Based on the division of a circle into 360° , the Greek Hipparchus, and later Claudius Ptolemy and Menelaus of Alexandria, carried out laborious calculations of the lengths of chords in the circle, corresponding to a range of angles.

Indian and Arab mathematicians also developed a passionate interest in astronomical calculations. In the tireless search for better instruments for calculating the geometries of a sphere – in order to describe the celestial sphere – the wise men of these Eastern cultures developed their own trigonometric tables.

Over the centuries, they undertook this painstaking work, gradually improving the values of the trigonometric tables. The fruits of this labour went on to influence some of the most outstanding and celebrated names in the history of mathematics, such as Leonardo de Pisa, also known as Fibonacci (1170–1250), Johann Müller or Regiomontanus (1436–1476), François Viète (1540–1603), and Nicolaus Copernicus (1473–1543) himself.

Georg Joachim Rheticus (1514–1576), a student of Copernicus, is credited with the first explicit definition of the trigonometric functions, not in relation to the chords of a circle, but as ratios between the sides of a right-angled triangle. The development of analytical geometry occurred much later, with the arrival of Cartesian coordinates and the study of curves and functions. It was then that the functions $\sin x$, $\cos x$, $\tan x$... became the object of detailed mathematical study.

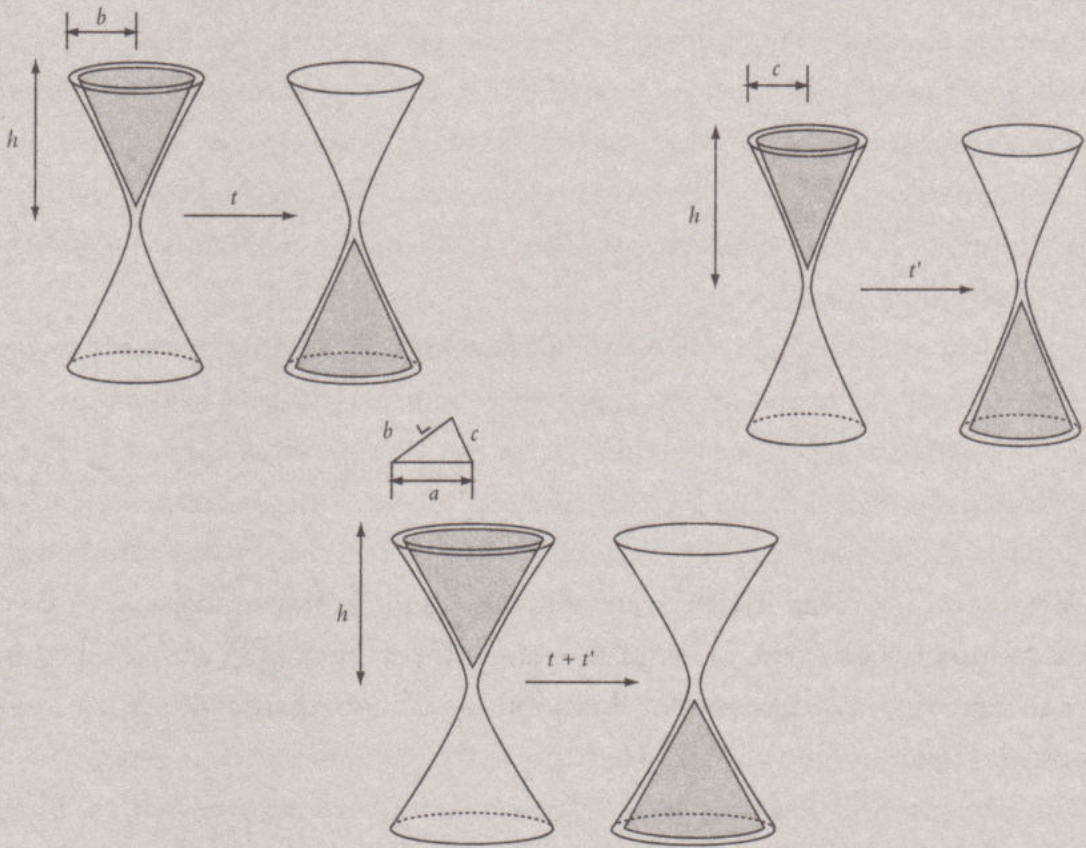
Trigonometry arose out of an interest in geometry and applications in architecture and surveying. A quotation by the American mathematician Elisha Scott Loomis sums it up: “Trigonometry is possible because Pythagoras’ theorem exists.”

The division of land into triangles (triangulation) has always been an essential method for the calculation of areas. The development of modern technology has confirmed the efficiency of this method. All triangles can always be divided into two right-angled triangles and these make it possible to calculate heights or distances based on the measurements of certain sides and angles. Taking a careful look at the figures and comparing them with the definitions of sine and cosine, it is possible to verify extremely useful properties. For example, $b = a \sin B$. This means that if the angle B is calculated to have value a , thanks to the trigonometric tables, we know the value of the side b . With the help of measuring tapes and theodolites, which give us

precise measurements for lengths and angles, this becomes a magic wand for carrying out all sorts of technical measurements. In fact the Egyptian surveyors were really priests and their calculations of territory, today indubitably secular, were considered as almost mystic, arousing nothing short of reverence from the farmers.

PYTHAGORAS' THEOREM FOR ADDING TIMES

Pythagoras' theorem can be used to add up squares, however suppose we have two hourglasses of equal height and want to make another that marks a period of time that is the sum of the first two glasses... Pythagoras' theorem shows us how to do it!



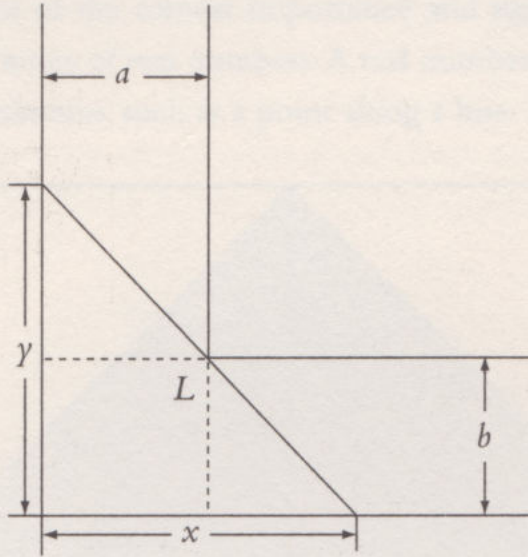
$$v_T = \frac{1}{3}\pi \cdot b^2 \cdot h + \frac{1}{3}\pi \cdot c^2 \cdot h = \frac{1}{3}\pi \cdot h \cdot (b^2 + c^2) = \frac{1}{3}\pi \cdot h \cdot a^2$$

- The radius a of the large hourglass must be the hypotenuse of the triangle with catheti b and c , which are the radii of the first and second hourglasses.

Everyday applications: moving furniture with Pythagoras' theorem

The Pythagorean equation is directly or indirectly present in dozens of everyday situations. For example, when installing a stair or ramp, or ensuring that two surfaces are exactly perpendicular; but also when making models, taking photographs or even just a simple photocopy – as we shall see in chapter 3.

Let us pause for a second to consider a problem that is common to many homes. It has been a long time since redecorating and it is decided to change the layout of the furniture between two rooms; we carry them through the hallway but come up against a tight corner. The hallway bends in the shape of a right angle. If only we could get the furniture around this turn.



What is the maximum size of an item of furniture that can pass through this space? As can be seen in the drawing, the key to turning in a corner is to invoke the Pythagorean equation $x^2 + y^2 = L^2$. Additionally, this figure also allows us to show proportionality:

$$\frac{b}{y} = \frac{(x-a)}{x}.$$

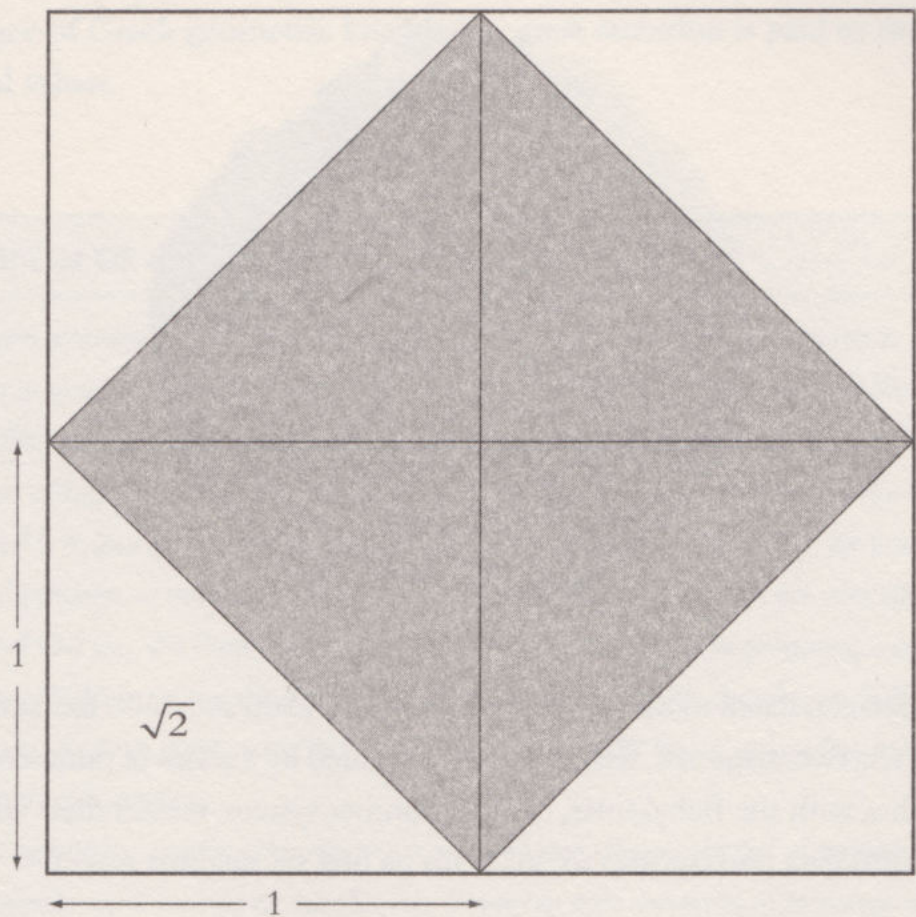
In chapter 4, we will see how special roots arise in the solution of these problems.

Chapter 3

An invitation to explore $\sqrt{2}$

Number is the ruler of forms and ideas.
Pythagoras

The number $\sqrt{2}$ was the first irrational number to be discovered. This proved to be a scientific event of the utmost importance and signalled a centuries-long investigation into the limits of real numbers. A real number is one that represents a distinct place in a continuum, such as a point along a line.

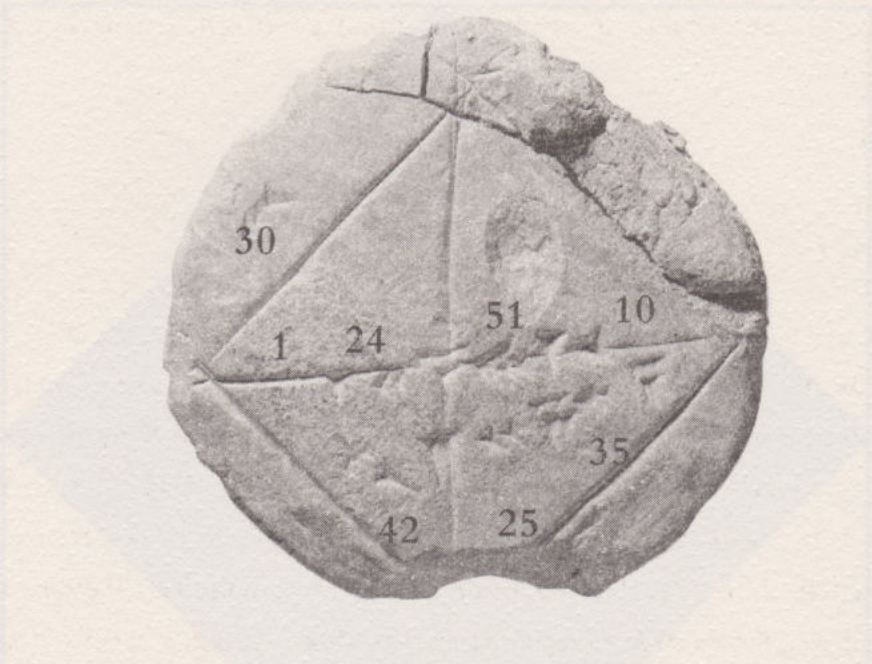


It is possible to find $\sqrt{2}$ with a simple sketch of a square. Take four squares with sides of 1 and draw in their four diagonals (as shown in the diagram). This gives us

another internal square (shaded here) that occupies half of the area of the four squares combined – with sides of length 2. The area of the large square is 4 (2×2) so the area of the shaded square is half this, 2, which means that the side of the shaded square multiplied by itself should be 2. Therefore, we have already obtained the square root of 2 (nowadays written as $\sqrt{2}$). The apparent simplicity of this operation belies the logical and computational problems that the number would come to represent.

The history of $\sqrt{2}$ (from 1800 B.C. to the current day)

In the collection of Babylonian clay tablets found in the prestigious Yale University, the tablet numbered YBC 7289 is of great importance because, as well as the usual cuneiform impressions, a quick glance is enough for any modern observer to clearly identify the very same diagram.



This artefact is dated to the period from between 1800 and 1600 B.C. and shows a square with its two diagonals, which are accompanied by a series of numbers marked with punches with the Babylonian base 60 number system. Researchers affirm that this numerical data corresponds to $\sqrt{2}$ with its first six decimal places:

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.41421296.$$

Later, in the ancient Indian books called *Sullasutras* (800–200 B.C.) the following written definition of $\sqrt{2}$ can be found: “...the length of the side is increased by a third and this third by its quarter and one thirty-fourth of this quarter is subtracted”, which, numerically, is

$$1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} = \frac{577}{408} = 1.414215686.$$

As can be seen, the Babylonians, Indians and, of course, the Egyptians, knew about and used fractions. However, their use of them was exclusively practical. It was not until the development of Greek mathematics that scholars began to take into account a more theoretical (and even philosophical) aspect of numbers.

The discovery of $\sqrt{2}$ as irrational is attributed to the Pythagorean School and therefore this number is irredeemably linked to the proof of Pythagoras’ theorem. In Euclid’s *Elements of Geometry*, which, as has already been said, is the work that marks the apogee of Greek geometric knowledge, great attention is paid to the nature of irrational values.

THE BIRTH OF QUADRATIC EQUATIONS

Geometric problems with squares, rectangles, triangles and other basic shapes led to the development of quadratic equations. In around 2000 B.C. in Babylonia there were already solutions to specific equations such as $x^2 + x = 3/4$ (tablet BM13901). The kind of real-world problem that lead to such equations can be expressed as: “If the area of the rectangle is $xy = 1$ and the perimeter is 4, $2x + 2y = 4$, the values of x and y are...”. On the Susa tablets, for example, the areas and squares of the sides of 3, 4, 5, 6 and 7-sided regular polygons are compared.

In around 628 B.C., the Hindu Brahmagupta solved $x^2 - 10x = -9$, also proposing a description of the method for solving this type of equation. Later, in the 9th century, we find Musa-al-Khwarizmi, an Arab mathematician who solved: $x^2 + 10x = 39$ and is responsible for founding the discipline of algebra.

Algebra replaced a notation convention that was very long winded and full of rhetoric, including long descriptions of how to do calculations. It was not until the times of Diofanto, in ancient Greece, when abbreviations were first used. Algebraic notation, with symbols like those we use here, was not developed until the 15th century.

The result is that there have been many attempts to approximate $\sqrt{2}$ using fractions. Analytic expressions have also been used to try to achieve this in a bold leap from the geometric world to that of arithmetic. It should be taken into account that $\sqrt{2}$ is intimately linked to trigonometry, since for 45° or $\pi/4$ radians we get

$$\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Therefore, powerful methods of calculating decimals are derived from knowledge of the trigonometric functions sine and cosine $\sqrt{2}$.

$\sqrt{2}$ is not only algebraic because it is the positive solution to the polynomial equation

$$x^2 - 2 = 0,$$

but also because, like all square roots, it can form part of the solutions to the quadratic equation:

$$ax^2 + bx + c = 0$$

with its two solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which gives the square root at the intersection points of parabola $y = ax^2 + bx + c$ with the x axis.

Fractional approximations of $\sqrt{2}$

A square with sides of 5 has a diagonal length of $5\sqrt{2}$ and as $(5\sqrt{2})^2 = 50$ is a number close to the actual square $49 = 7^2$, this gives $5\sqrt{2} \approx 7$, or $\sqrt{2} \approx 7/5$.

Therefore:

$$\left(\frac{7}{5}\right)^2 = \frac{49}{25} = \frac{50-1}{25} = 2 - \frac{1}{25} = 2 - 0.04.$$

This shows that there is a very small difference between a square with sides of $\sqrt{2}$ and one with sides of $7/5$. The fraction $7/5$ forms part of the series

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \dots$$

which is a good fractional approximations of $\sqrt{2}$.

It can be proven that some factors approximate to below the true value while others are above it:

$$\frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \frac{239}{169} < \sqrt{2} < \frac{577}{408} < \frac{99}{70} < \frac{17}{12} < \frac{3}{2}.$$

99/70 is a very close approximation:

$$\left(\frac{99}{70}\right)^2 = \frac{9,801}{4,900} = \frac{9,800}{4,900} + \frac{1}{4,900} = 2 + \frac{1}{70^2} \quad \text{and} \quad \frac{99}{70} - \sqrt{2} < 0.00025.$$

REPRESENTATIONS OF $\sqrt{2}$

$\sqrt{2}$ can be calculated with continued fractions:

$$\sqrt{2} = 1 + \left(1 / (2 + (1 / (2 + \dots)))\right).$$

Or as an infinite product:

$$\sqrt{2} = \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{38}\right) \left(1 - \frac{1}{100}\right) \dots$$

$$\sqrt{2} = \left(\frac{2 \cdot 2}{1 \cdot 3}\right) \left(\frac{6 \cdot 6}{5 \cdot 7}\right) \left(\frac{10 \cdot 10}{9 \cdot 11}\right) \left(\frac{14 \cdot 14}{13 \cdot 15}\right) \dots$$

$$\sqrt{2} = \left(1 + \frac{1}{1}\right) \left(1 - \frac{1}{3}\right) \left(1 + \frac{1}{5}\right) \dots$$

The Taylor formulas for trigonometric functions give:

$$\sqrt{2} = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \dots$$

And the Euler methods establish the following series:

$$\sqrt{2} = \frac{1}{2} + \frac{3}{8} + \frac{15}{64} + \frac{35}{256} + \frac{315}{4096} + \frac{693}{16384} + \dots$$

But there are obviously fractions with large numerators and denominators that are also close to $\sqrt{2}$:

$$\frac{351504323792998568782913107692171764446862638891}{248551090970421189729469573372814871029093002629}$$

Record calculations of $\sqrt{2}$

One of the most popular algorithms for calculating approximations of $\sqrt{2}$ is the so-called Babylonian method, which consists of starting with a positive integer F_n and applying the recurrence formula

$$F_{n+1} = (F_n + 2 / F_n) / 2.$$

With this method, Yasumasa Kanada's team managed to calculate no less than 137,438,953,444 decimal places of $\sqrt{2}$. More recently, in February 2006, Shigeru Kondo set his computer (a 3.6 GHz PC, with 16GB of memory) to work for 13

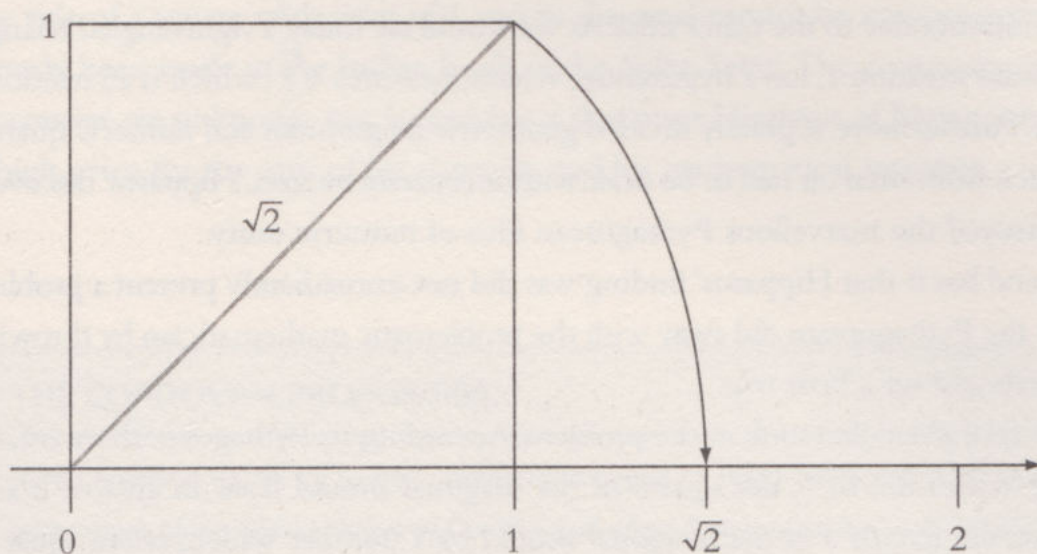
THE FIRST THOUSAND DECIMAL PLACES OF $\sqrt{2}$

$\sqrt{2} = 1.4142135623\ 7309504880\ 1688724209\ 6980785696\ 7187537694\ 8073176679$
 $7379907324\ 7846210703\ 8850387534\ 3276415727\ 3501384623\ 0912297024\ 9248360558$
 $5073721264\ 4121497099\ 9358314132\ 2266592750\ 5592755799\ 9505011527\ 8206057147$
 $0109559971\ 6059702745\ 3459686201\ 4728517418\ 6408891986\ 0955232923\ 0484308714$
 $3214508397\ 6260362799\ 5251407989\ 6872533965\ 4633180882\ 9640620615\ 2583523950$
 $5474575028\ 7759961729\ 8355752203\ 3753185701\ 1354374603\ 4084988471\ 6038689997$
 $0699004815\ 0305440277\ 9031645424\ 7823068492\ 9369186215\ 8057846311\ 1596668713$
 $0130156185\ 6898723723\ 5288509264\ 8612494977\ 1542183342\ 0428568606\ 0146824720$
 $7714358548\ 7415565706\ 9677653720\ 2264854470\ 1585880162\ 0758474922\ 6572260020$
 $8558446652\ 1458398893\ 9443709265\ 9180031138\ 8246468157\ 0826301005\ 9485870400$
 $3186480342\ 1948972782\ 9064104507\ 2636881313\ 7398552561\ 1732204024\ 5091227700$
 $2269411275\ 7362728049\ 5738108967\ 5040183698\ 6836845072\ 5799364729\ 0607629969$
 $4138047565\ 4823728997\ 1803268024\ 7442062926\ 9124859052\ 1810044598\ 4215059112$
 $0249441341\ 7285314781\ 0580360337\ 1077309182\ 8693147101\ 7111168391\ 6581726889$
 $4197587165\ 8215212822\ 9518488472$

days and 14 hours until obtaining 200,000,000,000 decimal places of $\sqrt{2}$. The only number which has received as much attention as $\sqrt{2}$ is pi.

These results were only possible in the digital age, because they are stored in computer memory. If an attempt was made to print Kondo's decimal places on normal sheets of paper, with 30 lines and 70 spaces on each (2,100 characters), nearly 100,000,000 sheets would be needed.

The surprising irrationality of $\sqrt{2}$



The Pythagoreans found powerful correlations between nature and mathematics. They discovered that the harmonious relationships between the musical notes correspond to quotients of integers. From this truth, they began to search for numerical proportions in all natural manifestations, convinced that reality as a whole could be explained with numbers. Everything is number, they asserted.

Pythagorean mathematicians consider that two magnitudes are commensurable if they are multiples of a third one, that is to say, if there is a common figure that allows the two magnitudes to have an integer (whole number) measurement. In other words, the Pythagoreans upheld the immovable principal that all numbers are a quotient of integers. The entire Pythagorean edifice was built on this basis, and thereafter, it sought to apply it to the whole universe in order to explain it in terms of numbers.

One day, Hippasus of Metapontum, an eminent member of the Pythagorean sect, set about playing with his master's theorem to calculate the diagonal of a

square. Squares were very simple figures and, strangely, nobody that Hippasus knew had bothered to calculate their diagonal and find out if its result would contribute anything to the great edifice of mathematics.

As a good follower of Pythagoras, Hippasus searched for universal demonstrations, and to make the problem of the square easy, he decided that its sides could be measured very simply, with a length of 1. The calculation was simple. The square was broken down into two triangles, and the master's theory was used to calculate the hypotenuse. The result was $\sqrt{2}$, a solution that completely devastated the most basic principles of Pythagoreanism, because it was not possible to make it a fraction.

Firstly, it demonstrated that the hypotenuse of a right-angled isosceles triangle is not commensurable to the other sides. As we would say today, a right-angled triangle whose sides measure 1, has a hypotenuse which measures $\sqrt{2}$, which is an irrational number. Furthermore, it greatly divided geometric magnitudes and numeric quantities, which from then on had to be dealt with as separate entities. Hippasus' discovery had destroyed the marvellous Pythagorean idea of numeric unity.

Legend has it that Hippasus' finding was did not immediately present a problem because the Pythagoreans did away with the problematic mathematician by throwing him overboard on a boat trip.

Let's take a detailed look at the problem. According to Pythagoras' theorem, for a square with sides of 1, the square of the diagonal should have an area of 2 and, therefore, the length d of that diagonal should be a number which, when squared, gives 2 (that is to say $d^2 = 2$). As we can see $\sqrt{2}$ was now on the scene.

$\sqrt{2}$ was the length of a segment that could easily be sketched from a square with the help of a ruler and a compass. Therefore, was it not reasonable to think that the side (1) and the diagonal ($\sqrt{2}$) of the square could also be measured in terms of a smaller unit u that was less than 1. Was it not reasonable to believe that the side and the diagonal of a square were surely commensurable? Yes, it was reasonable to think all of that. But, the side and the diagonal of a square just are not commensurable.

Presented in this way, the problem required repeating the common unit u with an integer number n of repetitions to measure the side $1 = nu$, and with another integer number of m repetitions, to find $\sqrt{2} = mu$. In other words

$$\sqrt{2} = \frac{\sqrt{2}}{1} = \frac{mu}{nu} = \frac{m}{n}.$$

So, the commensurability of 1 and $\sqrt{2}$ was reduced to the fraction m/n , which were both positive integers that were not commensurable.

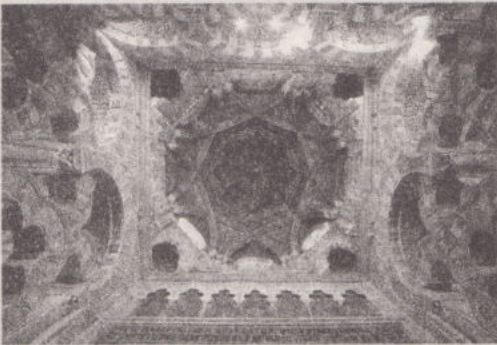
According to some historians, between 800 B.C. and 500 B.C., the observation that the side of a square with sides of 1 and its diagonal cannot be commensurable had already been made in the Indian book of the *Sulba Sutras*. The consequences for its discoverer are unknown, but legend has it that poor Hippasus of Metapontum paid a high price for the sum of his curiosity and his mathematical intuition.

THE CORDOVAN PROPORTION

When architect Rafael de La Hoz (1924-2000) began to study the relationship between proportions of the Great Mosque of Cordoba and other Arab designs in the Andalusian city, he discovered, to his surprise, the repeated presence of the number $1/\sqrt{2-\sqrt{2}}$ (or, 1.3065). The architect called this number "the Cordovan proportion". The number appears as a ratio between the radius of a circle and the side of a regular octagon drawn within that circle. As we know, Islam is iconoclastic but that does not mean that it completely renounces representation. In Muslim symbology divinity was represented by a square. Hence, the regular octagon, which is obtained from an intersection of two squares at 90° to one another, is

commonly found in domes and decorations.

The Cordovan proportion is present in all of these representations.



Octagonal base of a dome of the Great Mosque of Cordoba. The Cordovan proportion is the ratio between the side of this figure and the radius of the circle which it fits into.

The first proof of the irrationality of $\sqrt{2}$

During the Greek era, a beautiful proof that $\sqrt{2}$ could never be expressed as a fraction was found. The Platonic argument used the method of *reductio ad absurdum* in the following way:

“If $\sqrt{2}$ were a fraction, we could express it as a ratio of two integers a and b , and it would be $\sqrt{2} = a/b$, supposing that a and b do not have common factors. Squaring it would result in $2b^2 = a^2$, and given that the left-hand term $2b^2$ is even (divisible by 2), then 2 should divide into a^2 , which is only possible if a itself is even, that is to say, a would be expressed as $a=2c$. Therefore $2b^2 = a^2 = (2c)^2 = 4c^2$ y $b^2 = 2c^2$. As $2c^2$ is even, the factor 2 should divide into b^2 and, hence, b . But although it was supposed that a and b did not have common factors, they have proven to be even. This contradiction shows the impossibility of $\sqrt{2}$ being a fraction.”

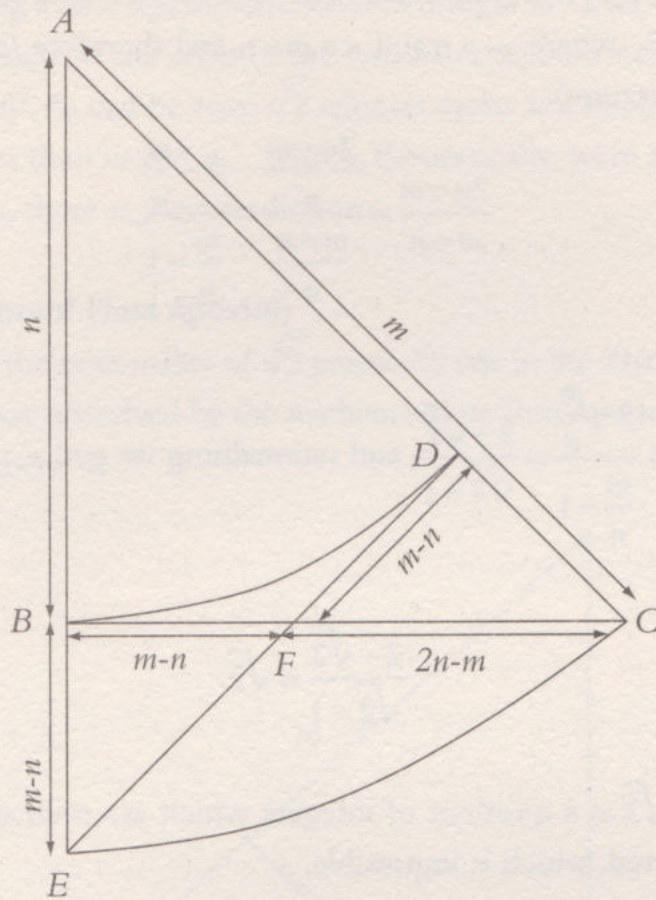
Further proof of irrationality

History has provided many arguments proving the irrationality of $\sqrt{2}$. Here follows some of the most interesting proofs.

WHY ARE SAUCEPANS NEVER SQUARE?

Saucepans are cylindrical. You will never see a prismatic saucepan with a square cross-section. It would quickly become obvious that a circle is the most suitable shape. As it does not have flat sides, it allows the contents to be stirred easily and is also easier to clean. On the other hand, a circular base allows the heat to be distributed symmetrically. But if you think about it, there is another, less obvious, but much more important explanation. The fundamental reason that saucepans are round is that the lid cannot fall inside. And although it may be hard to believe, we have to resort to $\sqrt{2}$ to explain this issue. For example, for a square saucepan with a square cross-section with 20 cm sides, the diagonal would be $20 \cdot \sqrt{2}$ cm, or, 28.28 cm. Therefore, the lid would have to be much bigger or it would certainly fall into the pan on occasion. However, this can never happen using circular equipment.

A geometric proof



If $\sqrt{2} > 1$ were a fraction m/n , m, n being the smallest integers (not having common factors) which give $\sqrt{2}$, given that $m^2 = 2n^2$, we would have a right-angled isosceles triangle ABC with sides of n and a hypotenuse of m . As indicated in the diagram, circular arc BD , arc EC and segments DE and BC are plotted with a ruler and compass. As EBF is an isosceles triangle and $BE = FB = m - n$, we get $FC = FE = n - (m - n) = 2n - m$. Then EBF gives a triangle with sides that are smaller than that with which we began, allowing $\sqrt{2}$ to be expressed as $(2n - m) / (m - n)$. This contradicts that n and m were the smallest numbers producing the fraction $\sqrt{2}$.

A proof with factors

If $\sqrt{2} = m/n$ were a fraction with integers m and n without common divisors, and we were to transform the expression by squaring it to eliminate the square root, we would get that $m^2 = 2n^2$. By writing m and n as products of powers of prime numbers, the value of $2n^2$ will always be an even expression, which means that m^2 must also be. This is absurd because we started with two numbers without common divisors – one of which must therefore be odd.

A proof with calculus (Miklós Laczkovich)

If $\sqrt{2} = m/n$, where m, n are the smallest integers which give that expression, we would get $m^2 = 2n^2$, where $m > n$ and $n > m - n$ and therefore (going back to the geometric demonstration):

$$\frac{2n-m}{m-n} = \frac{\frac{2n-m}{n}}{\frac{m-n}{n}} = \frac{2-\frac{m}{n}}{\frac{m}{n}-1}.$$

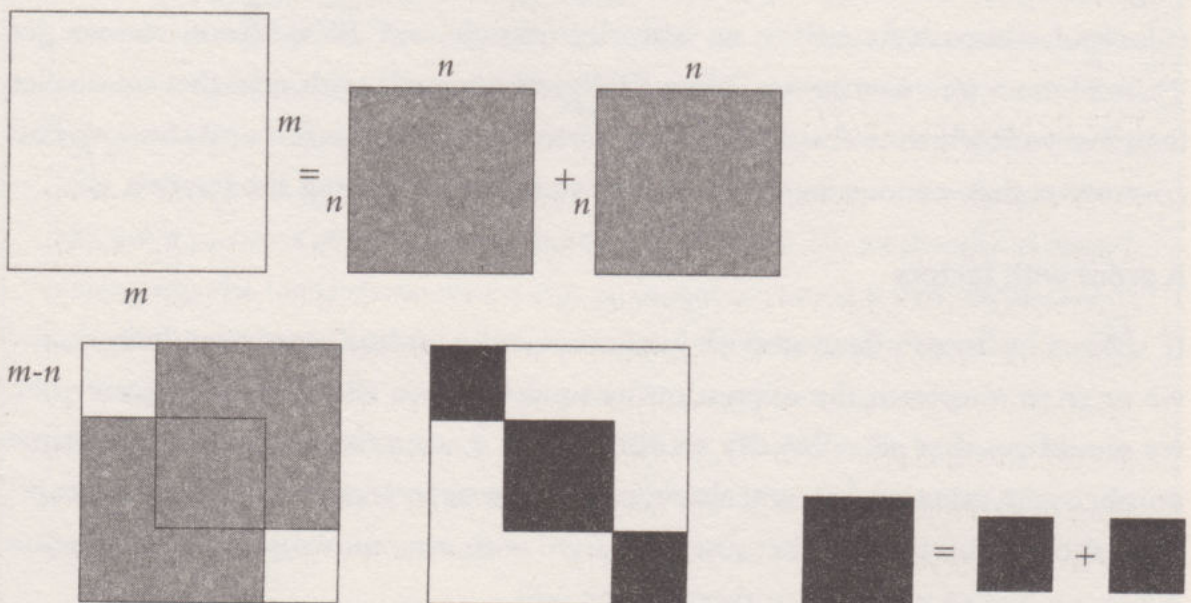
As $\frac{m}{n} = \sqrt{2}$ then $\frac{2-\frac{m}{n}}{\frac{m}{n}-1} = \frac{2-\sqrt{2}}{\sqrt{2}-1}$; and rationalising we get:

$$\frac{2-\sqrt{2}}{\sqrt{2}-1} = \sqrt{2}.$$

This expresses $\sqrt{2}$ as a quotient of integers which are even smaller than those with which we started, which is impossible.

An ingenious graphical proof (Alexander J. Hahn)

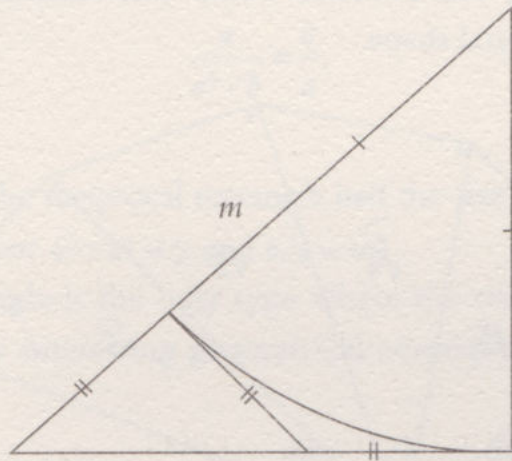
If $\sqrt{2}$ were a fraction, m/n would be $m^2 = 2n^2$, and the white square with sides of m would be equal in area to the two grey squares next to it n .



When the grey squares are placed in opposite corners of the white square, they overlap giving a new square with sides of integer $m-2(m-n) = 2n-m$, whose area must be the sum of the areas of the two black squares next to it $m-n$. Thus, $(2n-m)^2 = 2(m-n)^2$. As can be seen $\sqrt{2}$ appears again as a fraction $(2n-m)/(m-n)$ of integers smaller than m and n ... which, theoretically, were already the smallest ones! Once again, there is a contradiction.

A diagrammatic proof (Tom Apostol)

This final proof of the irrationality of $\sqrt{2}$ practically saw in the 21st century, as it is from the year 2000. It was conceived by the mathematician, Tom Apostol, who specialises in the theory of numbers.

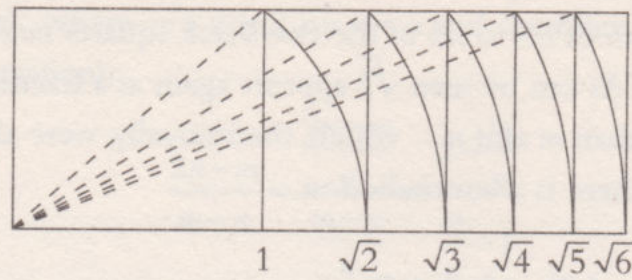


The smallest right-angled triangle with integer sides m, n, n where $m^2 = 2n^2$ allows another, even smaller triangle with integer sides to fit inside it. This contradicts the 'smallest' values that were supposed at the beginning.

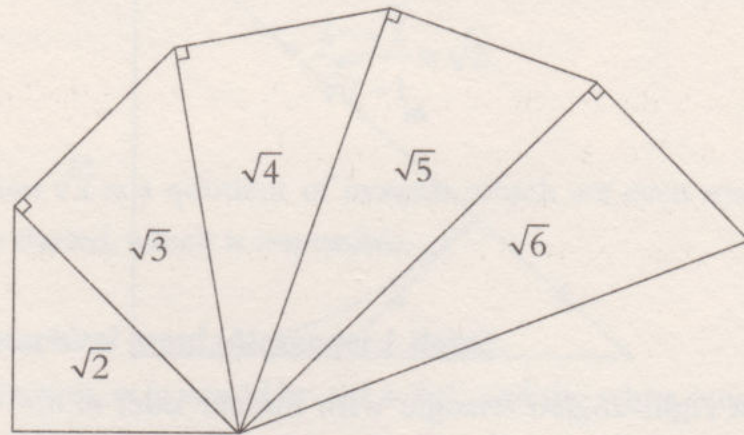
THE PROBLEM OF $2\sqrt{2}$

At the International Congress of Mathematics held in Paris in 1900, the influential German mathematician David Hilbert presented what he considered the contemporary boundaries of Mathematics and encouraged the professional community to go off and break through them. Hilbert proposed a list of 23 unresolved problems which, in his opinion, were of the utmost importance. Among them was Fermat's Last Theorem, but also $\sqrt{2}$. The question put forward regarding it was: Is $2\sqrt{2}$ a transcendental number? That is to say, it cannot be solved to a polynomial expression with integer coefficients equal to zero. Hilbert sensed that the answer was yes, that $2\sqrt{2}$ was transcendental. In 1930, thirty years later, it was proven.

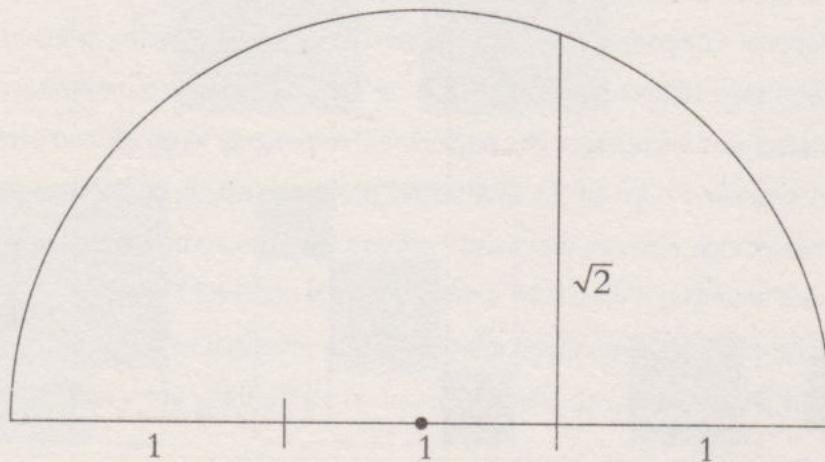
Geometric sketches of $\sqrt{2}$



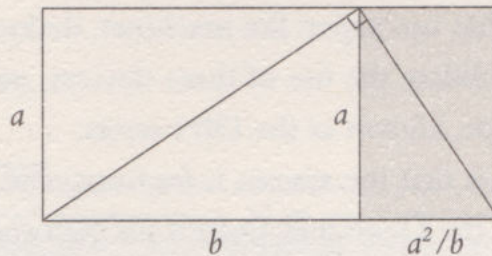
The first geometric sketch of $\sqrt{2}$ is the diagonal of a single-unit square. In the diagram it can be seen that, by locating $\sqrt{2}$ with a ruler and compass, we get a rectangle with sides of 1 and $\sqrt{2}$, the diagonal of which is $\sqrt{3}$. From here, all the segments appear one by one $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \dots$. The same series can be generated visually by drawing a spiral shape.



Another way of sketching $\sqrt{2}$ is with circumferences. By drawing a semicircle with a diameter of 3 and, at the point of the diameter that is 2 units from one side, we draw a perpendicular line, the length of this line will be $\sqrt{2}$.



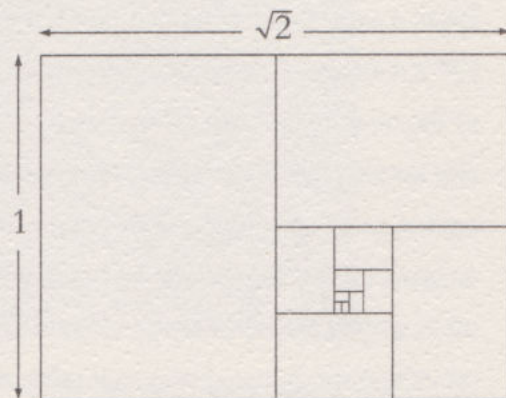
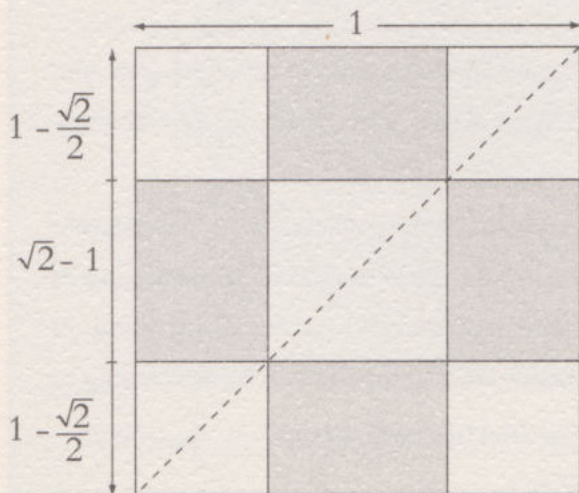
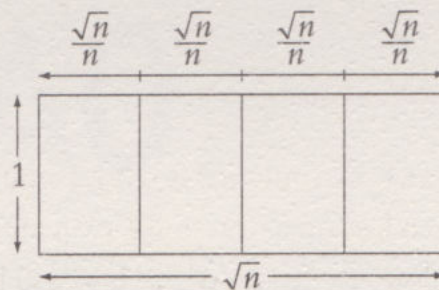
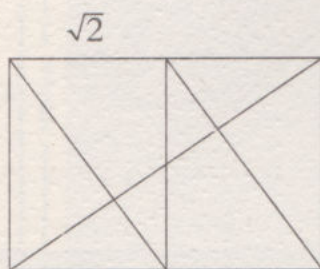
Given a rectangle $a \times b$, its diagonal can always be drawn. And the perpendicular to this diagonal can be drawn outside (or inside) from one vertex, making what is called a reciprocal rectangle. This new rectangle, as well as having the same shape, shares one side with the initial rectangle. Therefore, the ratio of its sides is identical a and a^2/b :



$$\frac{a}{a^2/b} = \frac{b}{a}.$$

But then, when is the reciprocal rectangle half the size of the initial rectangle? It must be $a^2/b = b/2$, or $b^2/a^2 = 2$ and $b/a = \sqrt{2}$.

The $\sqrt{2}$ ratio rectangle is the only type where the reciprocal is the half. The following figures show interesting geometrical consequences derived from this property.



DIN paper and photocopy formats

Many readers may still be familiar with the old paper formats, namely the foolscap (343×432 mm), small post (469×368 mm), legal (355×215 mm), small foolscap, etc.

Technological progress has forced typewriters into retirement and has created a new world of devices that use paper: fax machines, desktop printers and photocopyers. In order to consolidate the use of these devices, standard paper sizes have been introduced in Europe, known as the DIN series.

The interesting thing is that the system is far from new. The DIN system is an old German design from the Deutsches Institut für Normung, whose initials give us its name. The DIN was an institution dedicated to standardisation, which gave Walter Forstmann the job of drafting a regulation (DIN 476) for setting paper formats in 1922.

The International Organisation for Standardisation (ISO) included the German regulation in its own ISO 216 regulation. This organisation was established after the Second World War in order to promote international regulations in industry, commerce and communication, thus increasing guarantees of the quality and safety of products. Nowadays, some 160 countries form part of this organisation, which has headquarters in the Swiss city of Geneva.

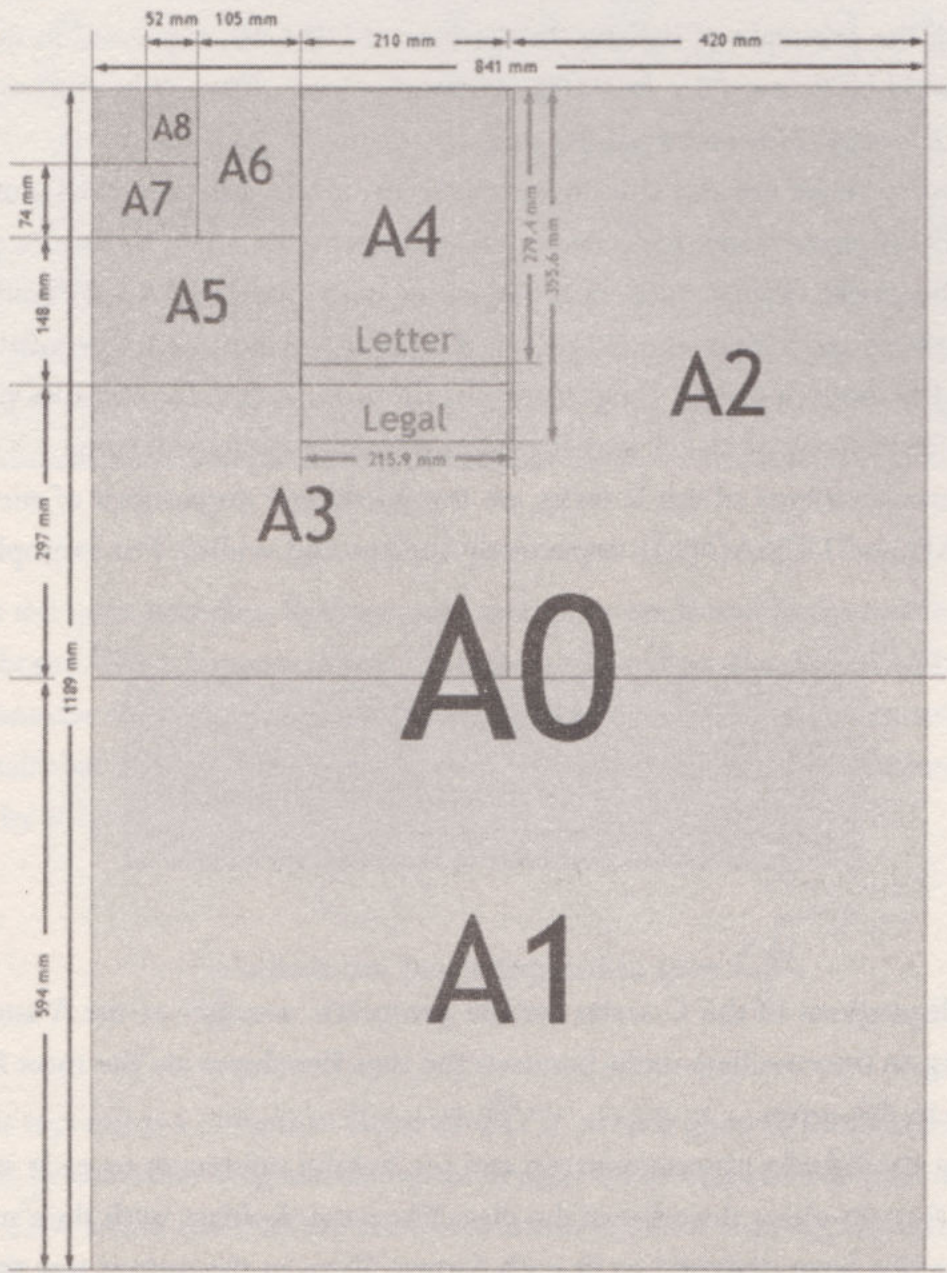
FROM DUPLICATING A SQUARE TO DUPLICATING A CUBE.

Given a square with sides of 1 unit is it possible to draw another square of twice the area with just a ruler and a compass? The answer is yes, since the diagonal of the square $\sqrt{2}$ is the side of the new square with an area of 2.

This simple problem was easily resolved, but then, in ancient Greece, the question that naturally follows came up: Could this be done with a cube? Given a cube with sides of 1, could side x of a cube of twice the volume be made with a ruler and a compass? This volume was $x^3 = 2$, in other words, they were trying to draw $\sqrt[3]{2}$, the cube root of 2. It took until the 19th century to prove that this drawing was impossible to perform with a ruler and compass.

Currently, there are few countries in the world that do not follow these formats: most notably the USA, Canada and some Latin American countries. Also, one country in Europe maintains a different system.

At the heart of this measurement unification lies a surprise that is completely unknown by the general public and is also very intriguing. The mathematical base of the DIN system is the $\sqrt{2}$ proportion and a surface area of 1m^2 .



DIN paper size formats from A0, with an area of 1 m^2 , to A8, not much bigger than a business card.

For rectangular sheets of paper $a \times b$ (where $a < b < 2a$), where the first sheet, DIN A0, has an area of 1 m^2 and each sheet divided in half along the long side b giving two parts with the same format, results in the ratio

$$\frac{b}{a} = \frac{a}{b/2},$$

in other words, $b^2 / a^2 = 2$, or equivalently $b / a = \sqrt{2}$.

As can be seen, $\sqrt{2}$ is the key to facilitating a coherent series of identical formats using the principle of halving. In the case of DIN A0, expressed in millimetres, it would be $b : a = \sqrt{2}$ y $ba = 1,000,000 \text{ mm}^2 = 1 \text{ m}^2$. This gives $a^2 \sqrt{2} = 10^6$, or $(10^6 \sqrt{2} / 2)^{1/2} = 841 \text{ mm}$ and $b = 1,189 \text{ mm}$.

Of course these are the final measurements of the cut sheets. Manufacturers of rolls of paper work with larger measurements in order to ensure perfect cuts. For example, $880 \times 1230 \text{ mm}$ pieces are made for DIN A0. As well as the A series, there are others, the B and the C. Their measurements in mm can be found in the table opposite. Despite all this theoretical development, in practice there are deviations of $\pm 1.5 \text{ mm}$ for measurements up to 150 mm .

The measurements of the B series are the geometric dimensions of the values of DIN AN and DIN A(N+1) respectively (height and width). For example DIN B4:

$$\begin{array}{l} \text{B4 (250} \times \text{353)} \quad 250 \text{ mm} = \sqrt{210 \times 297} \text{ mm, A4} \\ \quad \quad \quad \quad \quad 353 \text{ mm} = \sqrt{297 \times 420} \text{ mm, A3} \end{array}$$

The dimensions of the C series are the geometric averages of the A and B series, giving an intermediate series between the two. Envelopes are the most familiar members in this series.

Lifting the lid of a photocopier we can see helpful references to each series of the DIN formats along the edge of the glass. The paper holders, with their movable stops have also been designed to fit each format. Even in the output tray there are often adjustable notches or guides. The Pythagorean essence of the photocopier is evident in its design.

MEASUREMENTS OF THE DIN SERIES (IN MM)

Series A	A0	841 × 1189	Series B	B0	1000 × 1414	Series C	C0	917 × 1297
	A1	594 × 841		B1	707 × 1000		C1	648 × 917
	A2	420 × 594		B2	500 × 707		C2	458 × 648
	A3	297 × 420		B3	353 × 500		C3	324 × 158
	A4	210 × 297		B4	250 × 353		C4	229 × 324
	A5	148 × 210		B5	176 × 250		C5	162 × 229
	A6	105 × 148		B6	125 × 176		C6	114 × 162
	A7	74 × 105		B7	88 × 125		C7/6	81 × 162
	A8	52 × 74		B8	62 × 88		C7	81 × 114
	A9	37 × 52		B9	44 × 62		C8	57 × 81
	A10	26 × 37		B10	31 × 44		C9	40 × 57
							C10	28 × 40

But the most complex Pythagorean operations are found in the machine’s electronic brains. What happens when we increase or reduce the size of a photocopy? Let’s consider the case of a DIN A3 being reduced to DIN A4. As the area of DIN A4 is half that of DIN A3, the mathematical process for adjusting the lengths is to divide by $\sqrt{2}$, or

$$\frac{x}{\sqrt{2}} \text{ which can be rationalised to give } \frac{x\sqrt{2}}{2}.$$

This represents a reduction factor of $\sqrt{2} / 2$, which gives us the number 0.7071, or, 70%. In the same way, increasing the length of an original DIN A4 we multiply by $\sqrt{2} = 1.4142$ (as DIN A3 has twice the area). That is to say, it is increased by 141.42%.

In summary, each photocopy we make, as irrelevant as it may seem, puts the modest $\sqrt{2}$ into action in a significant series of mathematical calculations.

***f* numbers in photography**

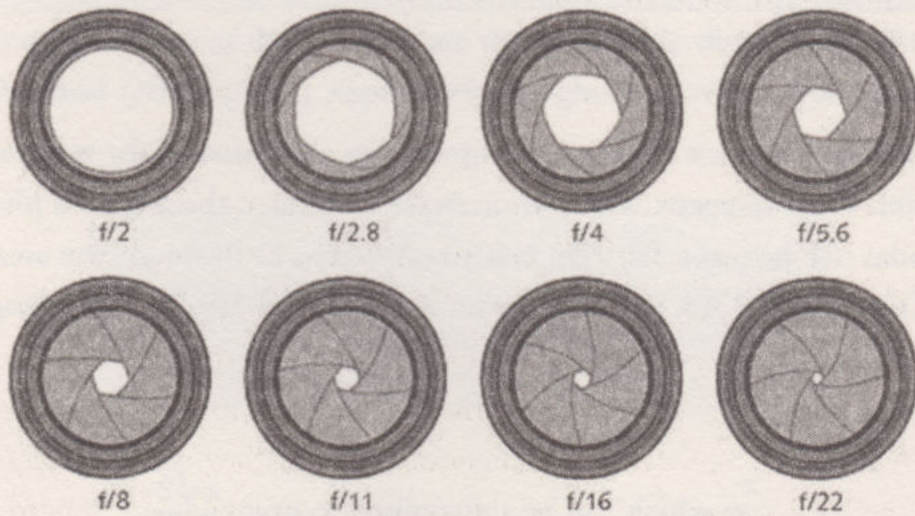
Let's consider a circle with a radius of R , the area of which will be πR^2 . If we want to create a circle of twice the area — $2\pi R^2$ — we need a radius of r , that is to say, $r = \sqrt{2}R$. Equally, to get half the area, we divide by $\sqrt{2}$. Therefore, the set of areas of double/half circles will amount to multiplying and dividing the radii or diameters by $\sqrt{2}$ each time. This brings into play the series of powers of $\sqrt{2}$:

$$\sqrt{2}, (\sqrt{2})^2, (\sqrt{2})^3, (\sqrt{2})^4, (\sqrt{2})^5, (\sqrt{2})^6, (\sqrt{2})^7, (\sqrt{2})^8 \dots$$

Which, rounded to 2 decimal places gives:

1.41; 2; 2.82; 4; 5.64; 8; 11.28; 16...

While the paper industry was designing its standard focusing on $\sqrt{2}$ and establishing proportions derived from dividing in papers half, the photography industry also found a motive to use this omnipresent number's proportions. $\sqrt{2}$ determines the apertures of camera lenses using the series of powers of $\sqrt{2}$ expressed in decimals. In photography they are called *f* numbers.



*Diaphragm apertures on an analogue camera
with the *f* numbers marked beneath.*

When we take a photograph in manual mode, we check the focus, the shutter speed and the exposure. We control the exposure of the camera by opening or closing the diaphragm (a small opening, which closes circularly). If the ambient light is very bright, we close the diaphragm, otherwise the photo will come out

A TUNNEL THROUGH A CUBE

Imagine a large wooden cube with sides of 1m in length. In it we want to make a hole through which another cube could pass. What is the largest cube that we could fit through? This interesting problem was solved by Dane Pieter Niewland. Niewland determined that the maximum size that could pass through the hole would be bigger than the original cube, with sides of 1.06066 metres, in other words, sides of length $3\sqrt{2}/4$.

over exposed. If the ambient light is very low, at night for example, we open the diaphragm, otherwise the photo will come out very dark. The diaphragm cannot simply be opened and closed on its own. The different positions should be used carefully as they need to be combined with other parameters, such as the shutter speed, which we mentioned above, in order to take a good photo. These positions are calculated with $\sqrt{2}$. When the diaphragm moves from one position to the next, the diameter is divided by $\sqrt{2}$, but the surface area of the aperture is halved, thus reducing the amount of light.

The f numbers for manual cameras were:

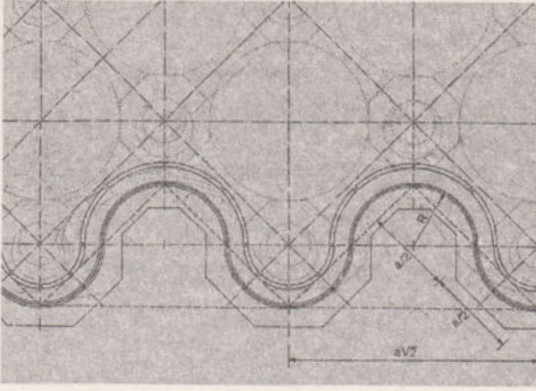
1.4, 2, 2.8, 4, 5.6, 8, 11, 16...

in other words the powers of $\sqrt{2}$ rounded to one decimal point, except for 11.28 which is somewhat drastically reduced to 11, without the very reasonable 11.2 even getting a look in.

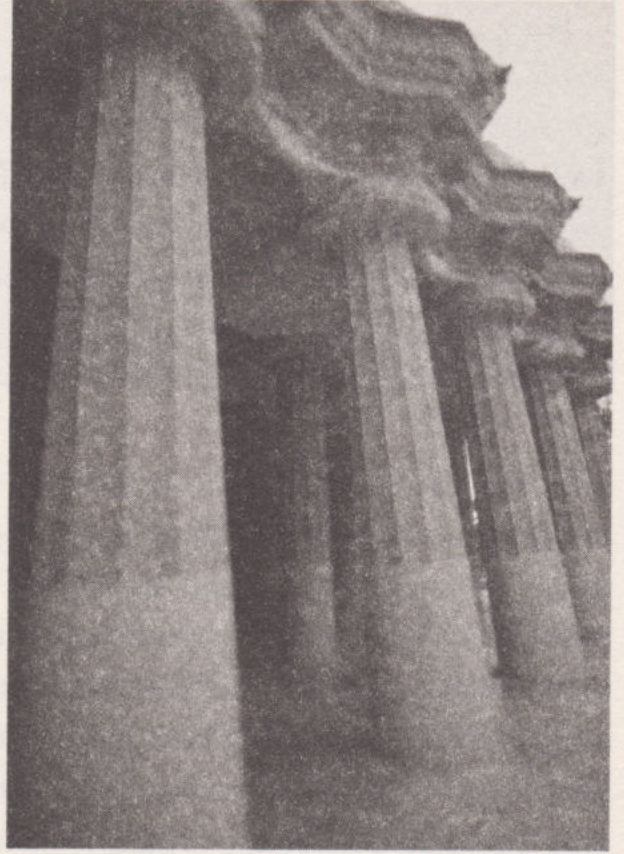
It is beginning to be a task of archaeological proportions to find those old film rolls with instructions that take us back to another age: "If it is cloudy use $f8$, if it is sunny use $f16$..." At the time they seemed straightforward, but they were actually just easy ways of saying: "What power of $\sqrt{2}$ would you prefer for taking your photo?"

$\sqrt{2}$ in Gaudí's Parque Güell

The square root of two as the diagonal of a unit square will always be present in all squares that are sketched, made or simply imagined. $\sqrt{2}$ is in tables, paintings, books; in everything that adapts square shapes or formats using $\sqrt{2}$ as a ratio. Perhaps it is not the most transcendental number, but $\sqrt{2}$ certainly holds a place of honour in art, architecture and design.



The upper ends of the columns which support the winding terrace of Parque Güell, while apparently irregular, are positioned in a perfect grid, as can be seen on the drawing.



We can see an example with a closer look at Parque Güell, designed by Barcelona's great architect, Antoni Gaudí. The architect created a series of unique spaces, which he perfectly integrated into the nature of the park. He erected a terrace, supported by immense Greek-style columns, as the centre point of the park. In it we can find the famous concave-convex bench decorated with a mosaic of brightly-coloured ceramic, whose trick of regularity and irregularity immediately catches the eye. Once again, the square hides a mathematical secret based on $\sqrt{2}$.

Gaudí located the upper centres of the columns in a grid of squares. A visitor observing the decoration of the meandering bench or the inclination of the columns would attribute it to architectural genius removed from any kind of rational system. But that perception is completely false. The stability of the unit is ensured by perfect squared geometry. Antoni Gaudí knew how to combine the most geometrically pure layouts with the simplest and most beautiful decoration.

Chapter 4

Journey to the spiral of Theodorus

The number governs the universe.

Pythagoras

A well-known Walt Disney film echoes a recurring joke on square roots. In *Donald Duck in Mathmagic Land*, the friendly duck arrives in this surprising land and finds himself in a forest of trees, the roots of which are formed from squares.

No doubt the script writers were not looking for anything deeper than a play on words here. However, as we can see, the relationship between square roots and the geometric shape of the square is an extremely close one. Nor did the Disney writers make much of a mistake in making square roots appear square in nature.

All square roots, not just $\sqrt{2}$, are linked to the lengths and proportions of geometric shapes. Pythagoras' theorem applies to a great many shapes that appear in graphical representations, with $\sqrt{3}, \sqrt{5}, \sqrt{6}...$ being easy to spot. This is perhaps not especially strange since artists themselves have used these numbers, albeit indirectly, by following rules for harmonic proportions based on mathematics.

What is surprising however, is that Pythagorean analysis allows us to identify the presence of square roots in thousands of naturally occurring forms, and almost inconceivably they appear in mathematically perfect proportions. They can be found in

Frontispiece of Tartaglia's Nova Scientia, subtitled "The Garden of the Mathematical Arts", with Euclid as the gatekeeper of the enclosure.



minerals that grow according to polyhedral patterns, from the prism shapes of some crystals to the cubes of pyrite. The measurements or calculations of any shape that is composed of these shapes produce some amazing mathematical results.

However, what is even more spectacular is to verify the presence of $\sqrt{2}, \sqrt{3}, \sqrt{5} \dots$ in the plant world. It can be found in flowers or fruits that exhibit repeating growth patterns, such as the polygons of petals and the spiral distribution of seeds.

ROOTS AND EQUATIONS: A HISTORICAL NOTE

Roots are inextricably bound up with the solution of equations of the second, third, fourth degree, etc., a problem that is elegantly described and tackled today using algebra, but which throughout the course of history has often been approximated using geometry.

Third-degree equations

The general problem of third-degree equations was not tackled until the Renaissance. In Italy in the 16th century, a self-taught mathematician, Niccolò Fontana, known better as Tartaglia (1499-1557), discovered some master formulae for solving this type of problem and even came up with poems that expressed the solution using verse.

In spite of the fact that Gerolamo Cardano (1501-1576), another Italian mathematician to whom Tartaglia had revealed his secret, promised not to disclose his discoveries, he then published the solutions under his own name, causing a considerable scandal in scientific circles.

Fourth-degree equations

A pupil of Cardano, the mathematician Ludovico Ferrari (1522-1565), discovered the formulae for the roots of fourth-degree equations based on those for the second and the third.

Equations of other degrees

Where it had been possible to provide general formulae using basic roots and arithmetic operations up to the fourth degree, everything seemed to indicate that it would perhaps be possible to discover a general solution for any equation of degree n . It was certainly possible to solve individual cases ($x^7 = 8$, the solution is the seventh root of 8) but not in general.

Paolo Ruffini (1765-1822), from Valentano, Italy, developed the famous rule that bears his name, a systematic manner of checking if a number solves a polynomial equation. Years later, based on the results of his predecessors, such as Euler, Lagrange and even Ruffini himself, Niels Henrik Abel (1802-1829) and Évariste Galois (1811-1832) proved that, in general, from degree $n = 5$ it was not possible to find arithmetical formulae for a solution.



Three examples from the animal and plant kingdoms in which roots and mathematical series correspond with the Pythagorean theorem.

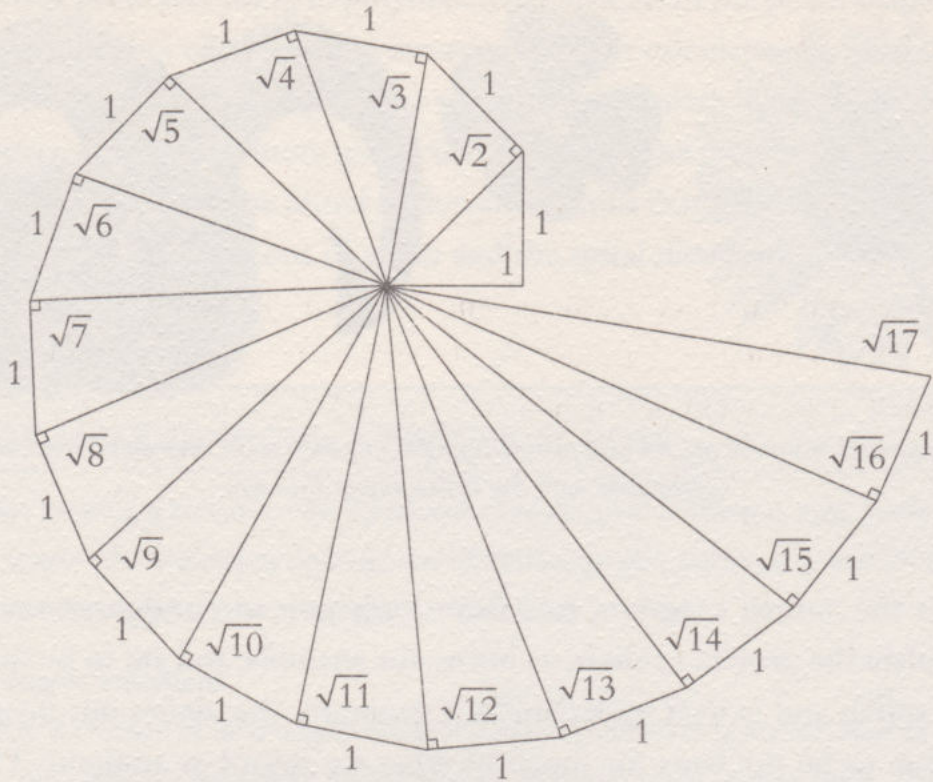
Nor is the animal kingdom free from Pythagorean configurations. Square roots regulate the growth of shells or horns, for example, and are to be found in a world of spirals and helixes underlain by a geometry that defies our imagination and appears to be the basis for much of what we regard as aesthetic. Theodore Andrea Cook, an art critic from the start of the 20th century, was the first to study this phenomenon. His work *Curves of Life* is considered a classic and the starting point for all subsequent analysis. The mathematics underlying these natural rules is reviewed next.

Dynamic proportions of $\sqrt{n}$

For a long time $\sqrt{2}$ was the only irrational number that was known. In his famous dialogue *Theaetetus*, Plato explains that his mathematics teacher, Theodorus of Cyrene, was the first to prove the irrationality of other square roots, sometime around 425 B.C. Theodorus considered the following lengths:

$$\sqrt{2}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}, \sqrt{17}.$$

Pythagoras' theorem allows for all the square roots of the natural numbers to be drawn in a spiral, the shape of which is determined by right-angled triangles. There is no proof that Theodorus of Cyrene drew the spiral. However history has given it his name, perhaps in acknowledgement of his proof of the irrational nature of lines of these lengths. Nowadays, the construction is rather grandly termed the Spiral of Theodorus.

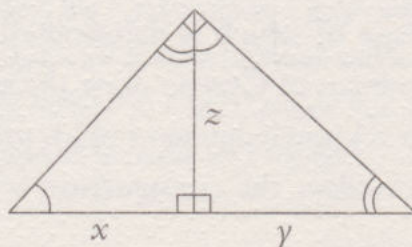


In the figure above, it is possible to see that $\sqrt{2}$ is the diagonal of a square with sides 1. However $\sqrt{3}$ is the diagonal of the rectangle with lengths 1 and $\sqrt{2}$ (which is equivalent to the hypotenuse of the right-angled triangle with sides 1 and $\sqrt{2}$). $\sqrt{4}$ is the hypotenuse of the catheti 1 and $\sqrt{3}$, and $\sqrt{5}$ is the hypotenuse of the catheti 1 and $\sqrt{4}$. Thus, the series of numbers $\sqrt{n}$ grows in a consecutive manner.

This recurring way of generating rectangles with lengths 1 and $\sqrt{n}$ means that the proportions $\sqrt{n}$ are described as dynamic proportions.

Playing with the dynamic proportions and applying the theorem of height it is possible to construct a reciprocal rectangle. But first, it is necessary to know the theorem.

Given a right-angled triangle with height z , the distance between the right angle and the hypotenuse divides the hypotenuse into two sections x, y .

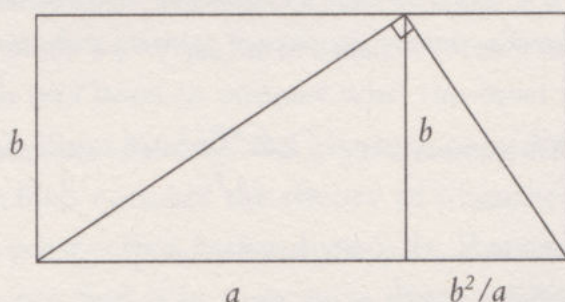


As such, the triangle is divided into two right-angled triangles with identical angles. Since they have the same shape, their sides are also proportional:

$$\frac{z}{y} = \frac{x}{z},$$

giving $z^2 = xy$ and thus $z = \sqrt{xy}$, the geometric mean of x and y .

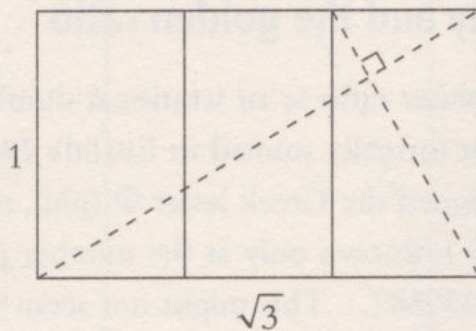
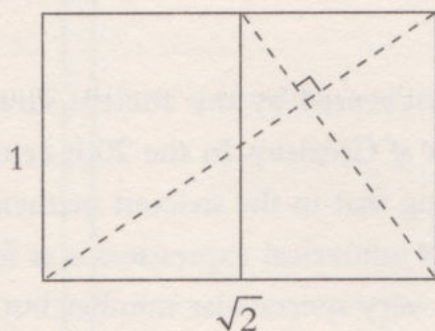
This result is known as the theorem of height. Let's now see its application to the construction of a reciprocal rectangle.



If we consider the reciprocal together with the rectangle with sides a, b with a right angle between diagonals, the sides of the new rectangle must be b and b^2/a . Thus, the proportion of the reciprocal rectangle is $b/(b^2/a) = a/b$, precisely the same as the rectangle we started with.

At what moment do n copies of the reciprocal rectangle cover the original one? This occurs when

$$n \cdot \frac{b^2}{a} = a.$$



But if $n = a^2/b^2$, then $\frac{a}{b} = \sqrt{n}$.

THE FORMULA OF HERO OF ALEXANDRIA AND $\sqrt{n}$

Hero of Alexandria was a unique figure who lived between 10 and 70 B.C. He was a geometer, mechanic, engineer, writer of encyclopaedias and a great inventor. In geometry, he gave his name to the formula for calculating the area A of a triangle based on its sides a , b , c . Calculating the semi-perimeter $s = (a+b+c)/2$ gives:

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

As we will see, this formula establishes square roots in the calculations of surfaces of any shape, since all that is necessary is to subdivide them into triangles. Hero also described a way to approximate $\sqrt{n}$, which is still used in computing to this day. Where $ab = n$, $\sqrt{n}$ is approximated using $(a+b)/2$.

As such, if $a_1 = a_1$ is an initial approximation, $a_2 = \frac{a_1 + n/a_1}{2}$ is an even better one,

and $a_3 = \frac{a_2 + n/a_2}{2}$ is better still, and so forth, ad infinitum.

For example, if $n=3$, and $a_1=2$, then $a_2 = \frac{2 + \frac{3}{2}}{2} = \frac{7}{4}$, $a_3 = \frac{\frac{7}{4} + \frac{3}{7/4}}{2}$, etc.

And thus we have demonstrated the marvellous property of dynamic proportions: $\sqrt{n}$ is the only function that makes it possible to divide a rectangle into n rectangular parts with the same shape. The DIN system for the standardisation of paper makes extensive use of them. The system is based on the application of the property $n=2$. For $n=3$ the property is used to make leaflets based on a sheet with proportions $\sqrt{3}$.

Beauty and the golden ratio

The golden ratio is an irrational number discovered by the ancient Greeks. It was first formally studied in Euclid's *Elements of Geometry*. In the 20th century, it was assigned the Greek letter Φ (phi), meaning that in the strictest mathematical terms it is known only as the number phi. Its numerical expression is as follows: 1.6180339887... This might not seem like a very spectacular number but when considered from the perspective of geometry its properties soon become clear. For example, when interpreted as a ratio, Euclid expresses it as such in the following words:

“It is said that a straight line is divided between the extreme and its mean when the whole line is to the greater segment what the greater is to the smaller.”

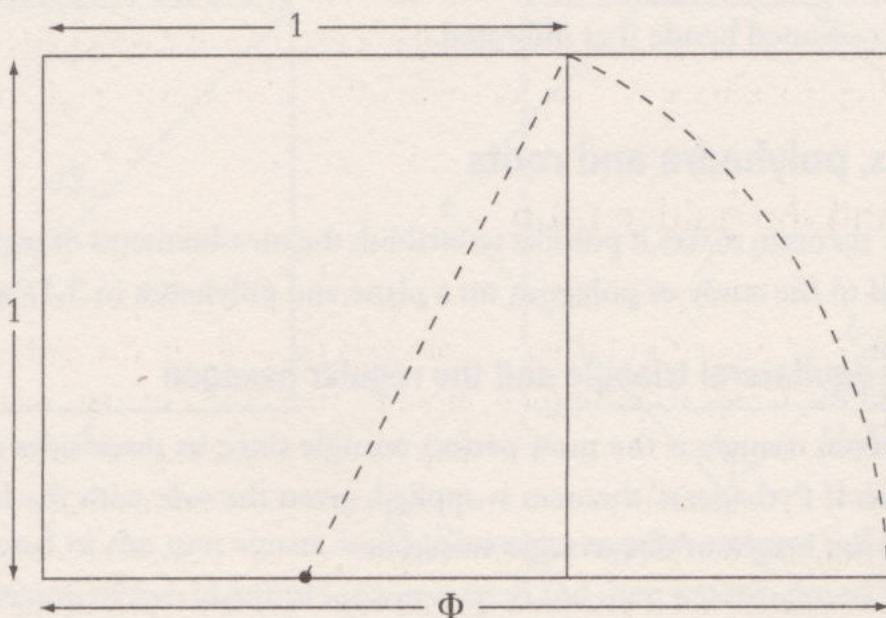
During the Renaissance, this ratio represented a motif of constant wonder. Leonardo da Vinci believed it to be the highest expression of harmony and adopted it as the rule of aesthetic beauty. It was da Vinci who christened phi and the proportion it represents as the ‘golden ratio’.

In 1509, a full treatise was published on the golden ratio. It was entitled *De Divina Proportione* (The Divine Proportion) and was written by Luca Pacioli, a Franciscan mathematician known for his pioneering work in the calculation of probabilities. Pacioli had been in contact with the most distinguished artists of his time, including da Vinci himself, and his work considers aspects of cosmology and mathematics, which connect the theory of Platonic solids to architectural styles and graphical perspective. Artists during the Renaissance used the number extensively, and we can find it in their most distinguished works.

The mythical golden ratio is expressed by the value:

$$\frac{1+\sqrt{5}}{2} \approx 1.618...$$

From Pacioli, it was said that any rectangle with the proportion Φ is golden. The following diagram shows a simple method of constructing rectangles with the golden ratio.

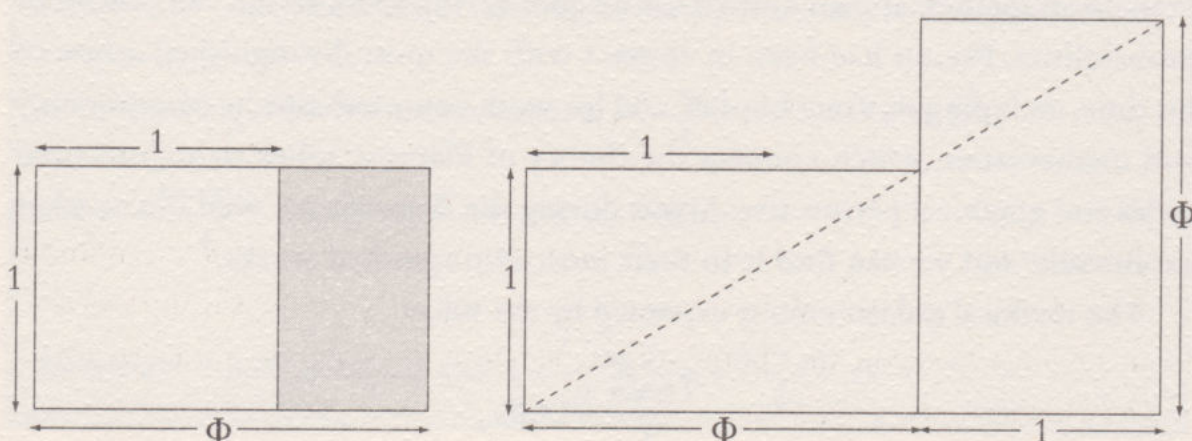


As can be seen, Pythagoras' theorem comes into play here. If the midpoint of a side of a square is joined to the opposite corner, this gives a segment of length

$$\sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2},$$

which, aligned with $\frac{1}{2}$, gives rise to $\frac{1}{2} + \frac{\sqrt{5}}{2}$, the golden ratio.

The following figures illustrate two notable geometric properties of golden rectangles.



The first shows that the golden rectangle is the only one that can be decomposed into a square and another rectangle that is also golden. In the second, the diagonal of one horizontal rectangle will pass through the upper right corner of a vertical rectangle positioned beside it as indicated.

Polygons, polyhedra and roots

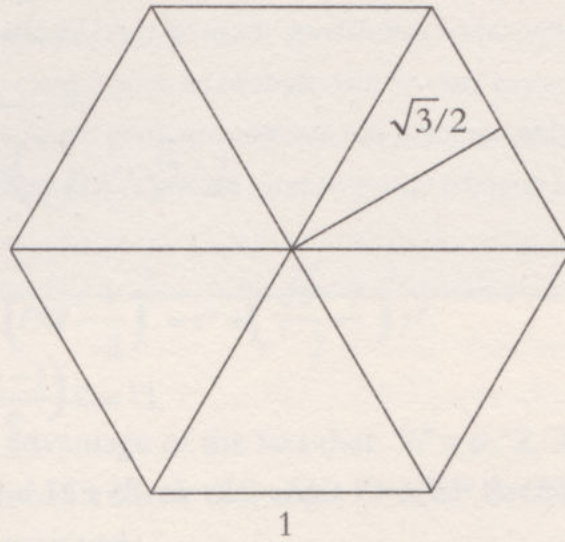
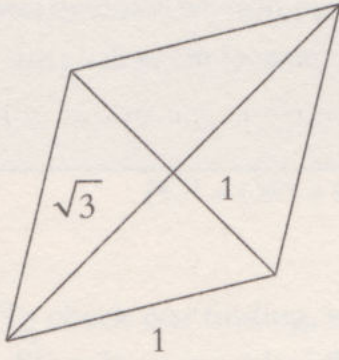
Pythagoras' theorem makes it possible to establish the measurements of segments that are essential to the study of polygons on a plane and polyhedra in 3-D space.

$\sqrt{3}$ in the equilateral triangle and the regular hexagon

The equilateral triangle is the most perfect triangle since its three sides and angles are all equal. If Pythagoras' theorem is applied, given the side with the length of a single unit, the height of this triangle would be

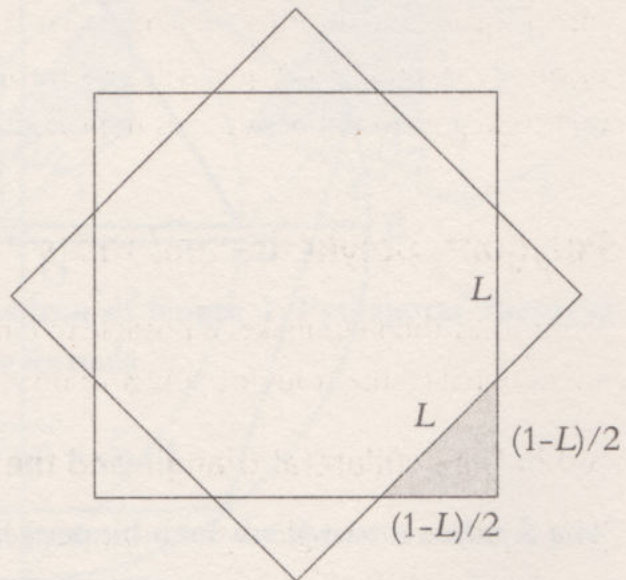
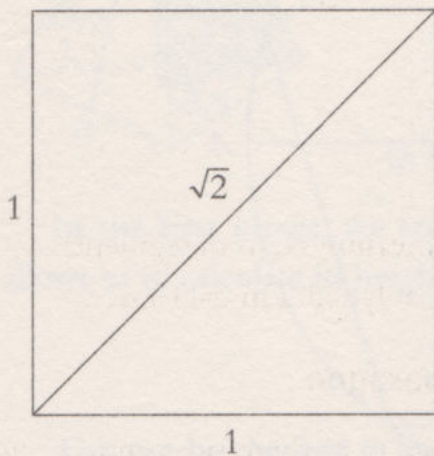
$$\sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}.$$

Thus, the area will be $\sqrt{3}/4$.



Joining two of these triangles by the base, it is possible to construct a rhombus with diagonals 1 and $\sqrt{3}$. With six repetitions of the same triangle, it is possible to form a regular hexagon, the apothem (distance from the centre to a side) of which is $\sqrt{3}/2$ and an area of $3\sqrt{3}/2$.

$\sqrt{2}$ in the square and the regular octagon



The diagonal of the unit square is $\sqrt{2}$. Since the regular octagon is formed from the intersection of two identical squares rotated and then superimposed, Pythagoras' theorem tells us that

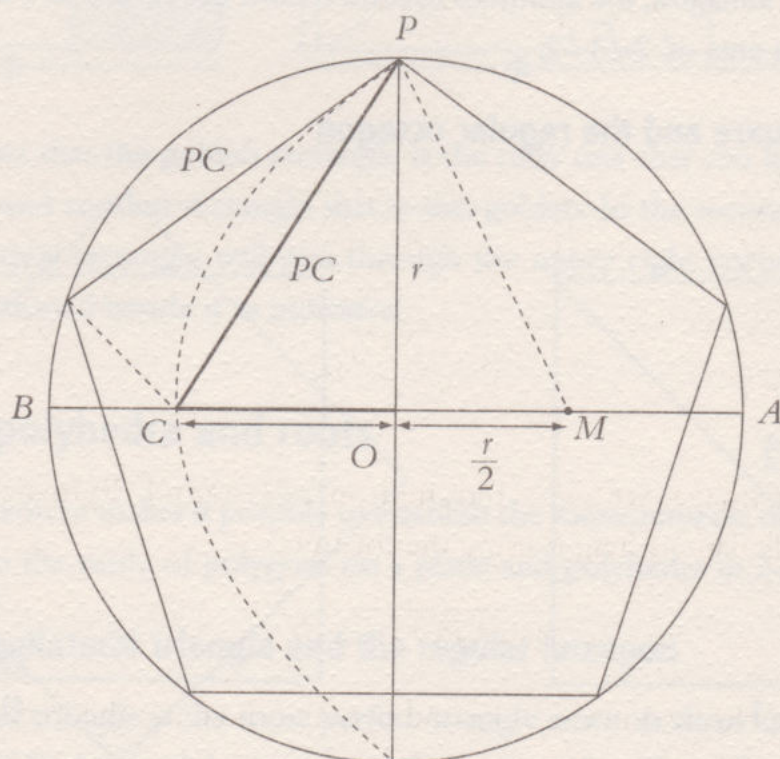
CASH OR CARD?

The majority of bank or business cards, parking tickets and even 16:9 television screens follow the golden ratio. It can be verified without needing to take measurements or carry out mathematical calculations. It can be checked playing with the shapes of two bank cards of the same size. All that needs to be done is place one of these horizontally and the other by its side, vertically, and line them up along the base. Trace the diagonal of the horizontal card and extend it until it ends up passing exactly through the right corner of the vertical card.

$$L^2 = 2 \left(\frac{1-L}{2} \right)^2,$$

so $2L^2 = (1-L)^2 = 1 + L^2 - 2L$, or $L^2 + 2L - 1 = 0$, which is the same as $L = \sqrt{2} - 1$.

$\sqrt{5}$ in the construction of a regular pentagon



A pencil, compass and ruler are all that are needed to produce the powerful harmony of a regular pentagon. However, without knowing about Pythagoras' theorem, the shape can appear askew.

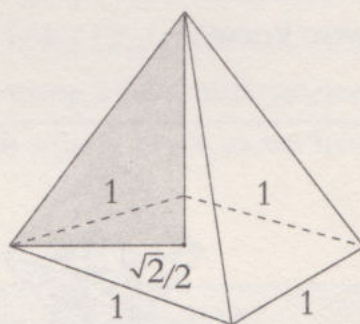
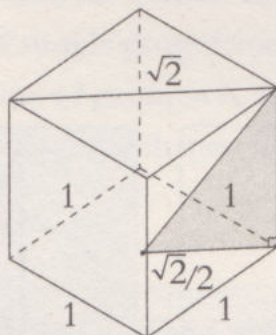
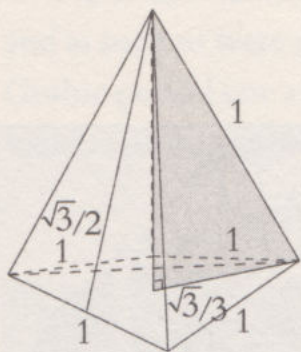
A circumference is drawn with centre O and radius r and the midpoint M of the radius OA is marked. With centre M and radius PM , an arc is drawn, the circumference of which cuts the radius OB at C . Thus, PC is the side of the regular pentagon contained in the original circumference. Why? To understand, it is necessary to apply the theorem twice.

$$PM = \sqrt{r^2 + \left(\frac{r}{2}\right)^2} = r\sqrt{5}/2,$$

$$PC^2 = OP^2 + OC^2 = r^2 + \left(PM - \frac{r}{2}\right)^2 = r^2 + \left(\frac{\sqrt{5}-1}{2}\right)^2 r^2.$$

To check our finding, we can take advantage of the fact that $36^\circ = \phi/2$. Thus, observing the triangle with sides r, r and PC , check that angle O is 72° or $360^\circ/5$. Thus, we have constructed a regular pentagon.

In the same way, it is possible to appreciate the presence of $\sqrt{2}, \sqrt{3}, \sqrt{5} \dots$ in all sorts of polyhedra. The roots are not only to be found on the faces of polygons, but also on diagonals, heights and projections.



In the first image, we see a tetrahedron of length 1. Pythagoras' theorem allows us to calculate its height using the formula

$$\sqrt{1^2 - (\sqrt{3}/3)^2} = \sqrt{6}/3.$$

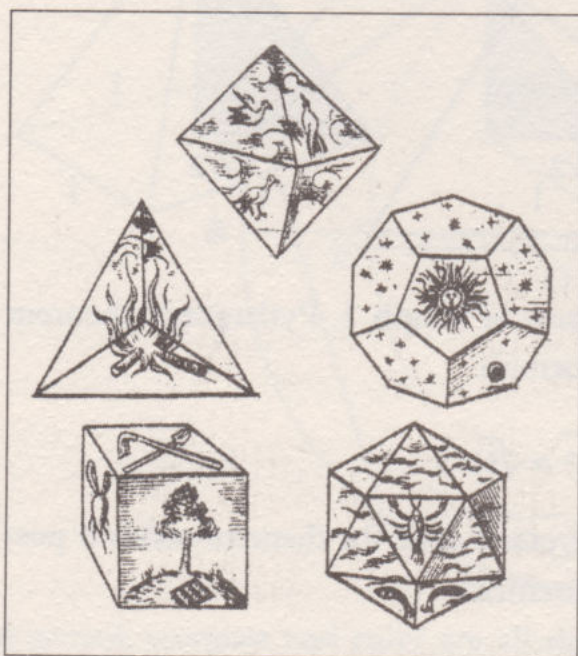
This can be checked in the cube and pyramid since the theorem makes it possible to calculate all types of interior measurements.

Pythagorean cosmogony with polyhedra

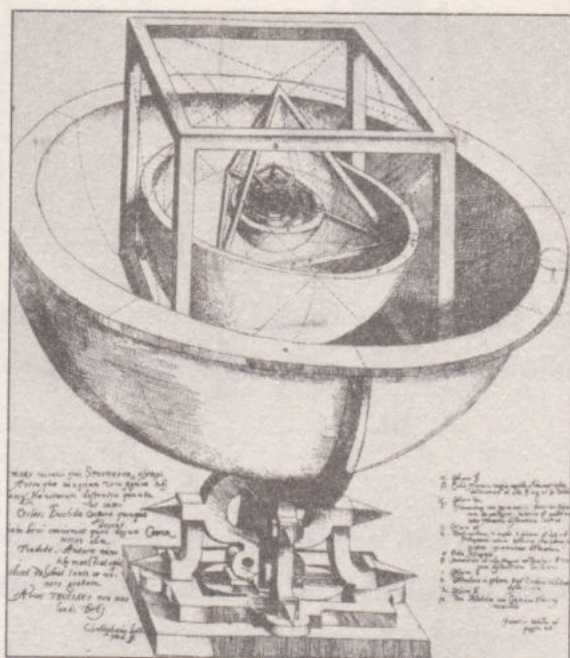
The writings of Philolaus of Croton, a pupil of Pythagoras, the Neoplatonist Proclus and the Byzantine Simplicius often made reference to the connection made by Pythagoreans between the four elements, fire, earth, air and water, and regular polyhedra, the tetrahedron, cube, octahedron and icosahedron, respectively. The series is completed by a fifth polyhedron, the dodecahedron, which symbolises the universe in its entirety. Its twelve faces are pentagonal, meaning that they contain the pentagonal star, a symbol of the Pythagorean identity. Although the Pythagorean school knew of these bodies and pointed to their cosmological symbolism, in reality it was Plato who was their biggest exponent. They are referred to as the *Platonic solids* to this very day.

In books XII and XIII of Euclid's *Elements*, he makes reference to the length of the edges of the five regular polyhedra. If they are considered as being contained within a sphere of radius 1, Euclid explains, they are the $2\sqrt{6}/3$ (tetrahedron) $\sqrt{2}$ (cube), $2\sqrt{3}/3$ (tetrahedron) $\sqrt{10(5-\sqrt{5})}/5$ (icosahedron) and $(\sqrt{15}-\sqrt{3})/3$ (dodecahedron).

These polyhedra fascinated artists such as Piero della Francesca, Luca Pacioli, Leonardo da Vinci and Albrecht Dürer, but they also led the astronomer Johannes Kepler to base an entire cosmology on them, in an era in which only six planets were known.



Illustrations from the work *Harmonice Mundi* by Kepler (1619).



Kepler's geometric cosmological model (*Mysterium Cosmographicum*, 1596).

Square roots, art and design

The reverence people have felt towards the occurrence of square roots in nature is frequently debated. The surprise is understandable. It has taken centuries for human intelligence to develop the fine mathematical instruments with which it tries to understand the world, while nature creates relationships and proportions that defy the most elaborate human intellectual constructions with incredible ease. However, scholars soon put this reverence aside and dedicated themselves to in-depth study of the mechanisms of these proportions and relationships. They aimed to use this knowledge to perfect a major capability: The construction of new worlds.

Painting, sculpture and architecture all make use of geometrical resources to create shapes on flat surfaces or in three dimensions. The series of roots $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5} \dots$ of course, the golden ratio, appear in the measurements and proportions of these figures.

It could be said that throughout history, aesthetic choices have kept pace with theoretical knowledge and technical abilities. Using geometric knowledge it has been possible to sketch out the general features of the history of art reinterpreted in terms of the development of mathematics.

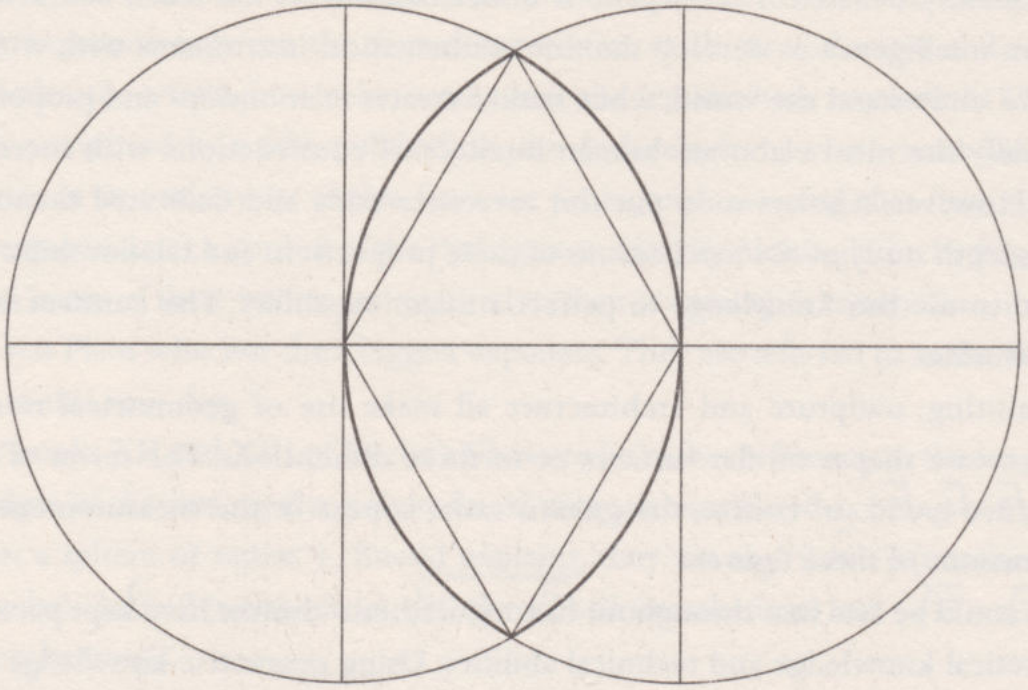
From ancient times to Medieval Rome, the use of grids was commonplace and as such so were the use of simple proportions ($1/3, 1/4, 1/2, 2/3, 3/4 \dots$). The Gothic period saw a flourishing of geometric drawings using a ruler and compass which saw an explosion of polygonal shapes, filled with square roots, as we have

SQUARE ROOTS AND THE PROBLEM OF LOOKING AT A STATUE

The German astronomer and mathematician Johann Müller (1436-1476), known as Regiomontanus, came up with a famous problem in the 15th century, which history has christened "The problem of Regiomontanus". Given a large statue that is on top of a large pedestal with the foot of the sculpture above the eye of the viewer, how near can the viewer stand and still see the whole statue – a position known as the maximum angle? The problem is considered as the first of its kind: the search for maximums.

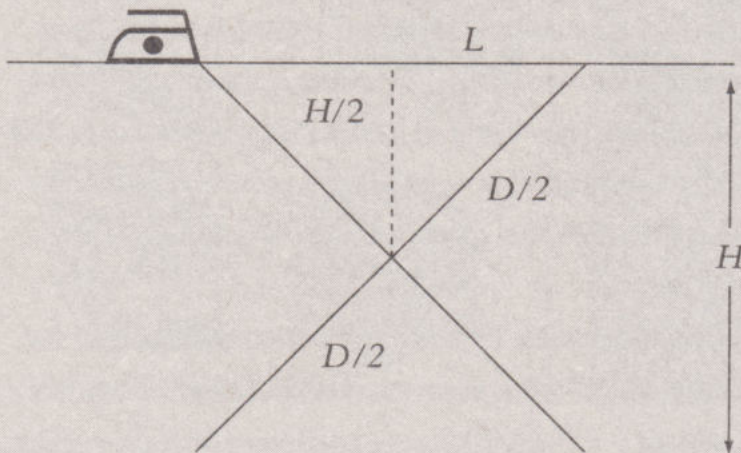
If the foot of the statue is a above the height of the viewer's eye and the highest point of the statue is b higher, the maximum angle of view occurs when the viewer is positioned precisely at a distance of $\sqrt{ab}$ from the pedestal.

seen. The typical Gothic arch is defined by two arcs, the circumferences of which are drawn from the opposite corners of an equilateral triangle $\sqrt{3}$.



PYTHAGORAS AND THE IRONING BOARD

The ironing board is an item that is both loved and hated. However, for our mathematical analysis, we are more interested in another of its defining properties: it can be folded. The ironing board has two extendible legs that allow it to be positioned at various heights. Who would believe that Pythagoras also appears here? If we look at the shapes made by the board and

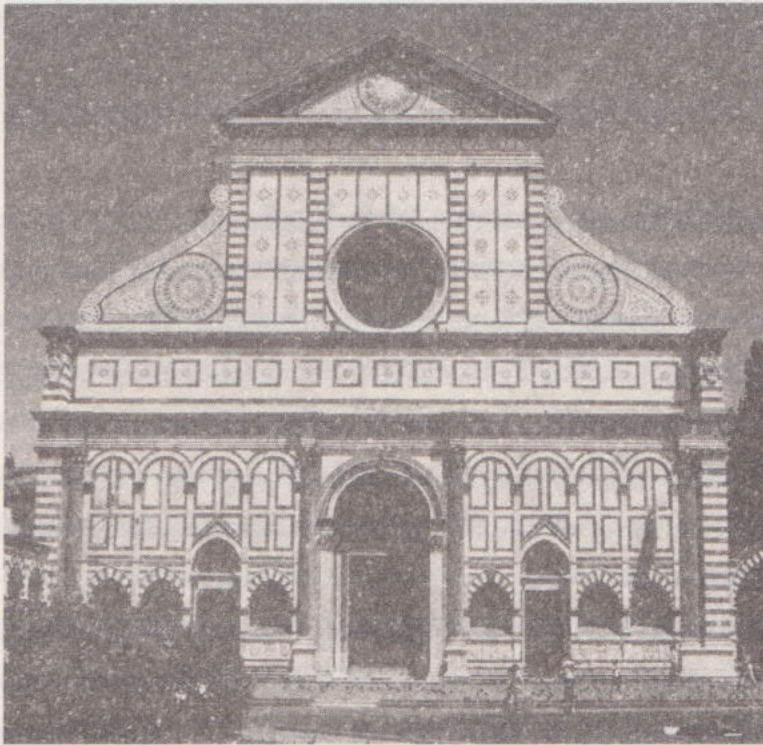


the two crossed legs, we can see that they form triangles. According to the drawing here and in line with Pythagoras' theorem, $(H/2)^2 + (D/2)^2 = L^2$, is the equation needed to figure out any of the measurements based on the other two.

Judging by their constructions, the Nasrid dynasty, the last Muslim dynasty to rule Granada in southern Spain, was extremely fond of geometry. During its rein, it built the Alhambra palace, a timeless jewel of Muslim art, which abounds in geometric forms not only in its decoration but also in the architecture of the building itself, with its octagonal-based domes. The Nasrid drawings that are based on squares contain $\sqrt{2}$ whereas in drawings with pentagons or decagons, $\sqrt{5}$, the golden ratio, appears.



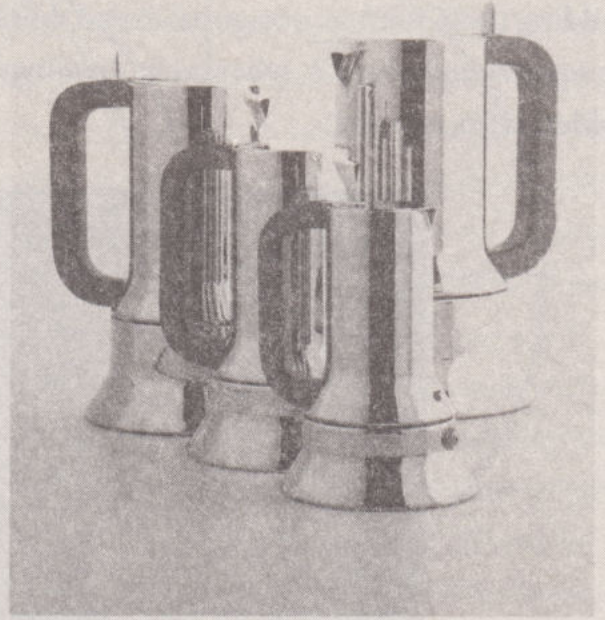
Nasrid geometrical design from the Alhambra.



Santa Maria Novella in Florence. The façade contains rectangles based on the golden ratio.

$\sqrt{3}$ AND THE ALESSI COFFEE MAKER

This celebrated coffee maker from the Italian design house Alessi, is preferred by geometers on account of its clever combination of cones and cylinders, and also, without doubt, for the way in which its metal handle doubles as a lock, avoiding the need for a plastic catch that may suffer from the heat. Aside from all of this, it holds a significant innovation: the coffee is served through a hole, which has the shape of a perfect equilateral triangle. If we consider this carefully, we realise that round shapes, such as formed by the mouths of bottles, are not the most

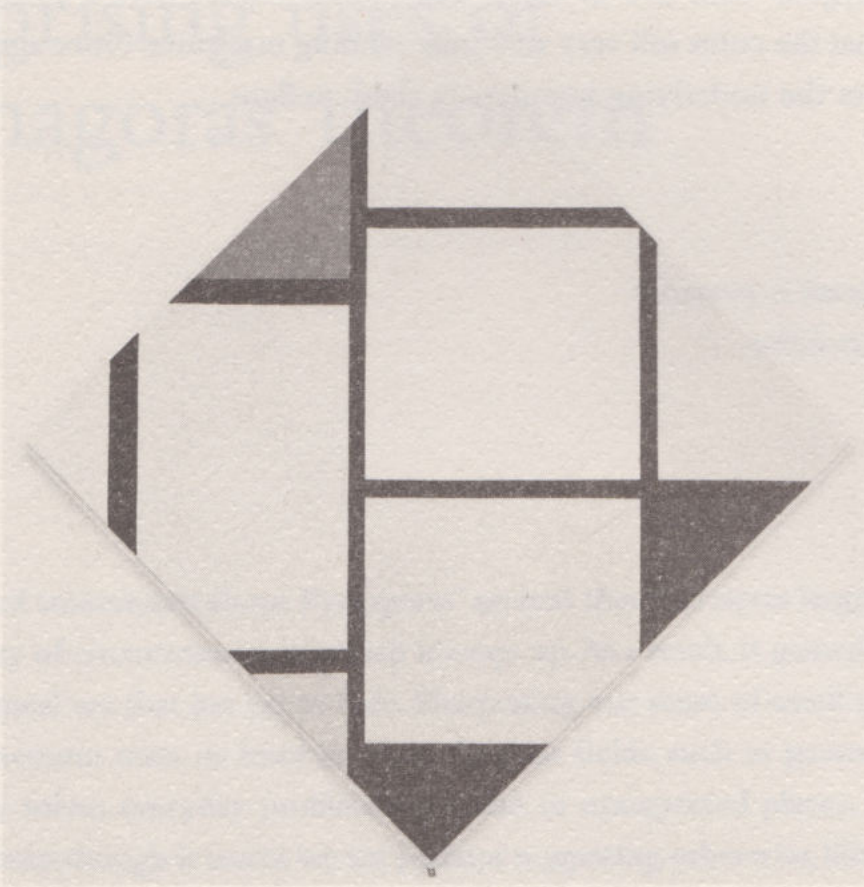


suitable for pouring liquids from a container to a glass, despite being very commonly used for this purpose. The equilateral triangle, associated with $\sqrt{3}$, allows the coffee to be poured in a perfectly controlled manner, with the spout vertex aimed directly at the cup.

The famous *Vitruvian Man*, drawn by da Vinci, attempts to reconcile the proportions of the human figure, showing a man contained within a square and also inside in a pentagon. Corbusier's response to this was the figurative human who appears in new take on units of measurements, the Modulor. In this system, the French architect returns to the classical idea of establishing a relationship between the proportions of a building and those of their human occupants. The ratios in the Modulor are extracted from the golden ratio or its powers, giving rise to a superhuman figure.

In spite of its Renaissance feel, the golden ratio still plays an important role in modern and contemporary architecture. And it goes further, while you might think that the removal of realism would result in a negation of all sense of proportion, this is precisely when the virtues of the purest geometric shapes and the associated proportions come to the fore. As an example, we can take any of the works

of Piet Mondrian (1872–1944), the Dutch avant-garde artist who was part of the De Stijl group, or the impossible-to-classify Paul Klee (1879–1940).



A painting by Mondrian in which the relationship between the work and $\sqrt{2}$ can be seen.

A really strange case is Reuleaux' triangle, made up of three equal arcs from circles that are traced from each corner of an equilateral triangle, passing through the other two corners. This figure hides $\sqrt{3}$, and in fact, it was previously used in Gothic masonry. It was the German engineer Franz Reuleaux (1829–1905) who discovered the dynamics of this shape. It is a constant width curve that runs like a circumference between two parallel lines, always touching both. It has been used in many common mechanisms, such as the Wankel engine used in many modern petrol-driven cars, or drill bits that create almost square holes when turning, amazing the observer. Some sweets have also made use of this shape to make them easier to roll around in the mouth. Today it is extremely common to come across dinner services that make use of this shape. The modern design has ended the tyranny of the circle for plates and glasses (although in this case, the only important thing is that the content stays where it is being drunk).

The appearance of these new and useful shapes are used to mint coins that are not round but made up of five or seven arcs with circumferences associated with regular polygons with five or seven sides. These curves have a constant width, meaning that the coins roll very well into vending machines. Once again, $\sqrt{5}$ and its group are the underlying numbers in these designs.

Chapter 5

Surprising uses of Pythagoras' theorem

*Geometry is knowledge of that
which exists eternally*
Pythagoras

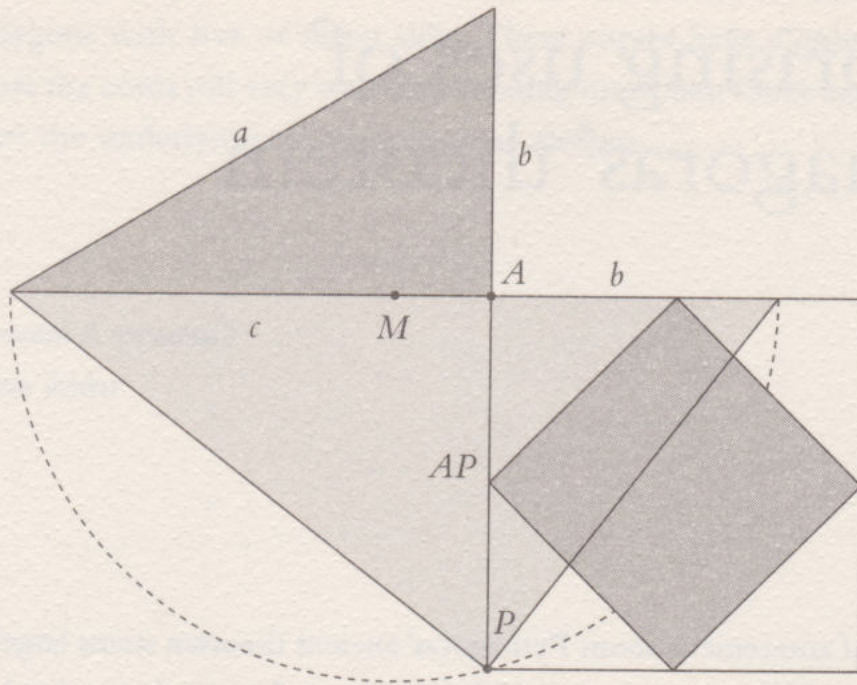
Any sense of amazement about Pythagoras' ancient theorem stems largely from the wide variety of circumstances in which it crops up. As a result, it gained a mystique in the Classical era that has yet to fade. Sharpening our sense of smell to follow its trail, the theorem rears its head in many familiar fields, such as geometry, where it elegantly solves everyday problems, but also in unexpected places, such as art. However, even though it seems we are perhaps suggesting otherwise, life is perfectly possible without knowledge of the theorem. Yet its presence in almost every corner of human intellect and scattered throughout nature cannot fail to captivate.

Squaring shapes

The relative simplicity of calculating the area of a square lead to attempts to solve the general problem of squaring other shapes. This involved finding a method for constructing a square with the same area as any given shape, using just a ruler and compass. If it were possible to draw these squares, this would give us a valuable graphical tool that enables us to solve the formidable problem of calculating surface areas. Given a linear measurement L (in whatever unit, a foot, hand, or metre...), it immediately becomes possible to obtain the measurement of a surface L^2 , which is formed by the area of the square with length L (foot squared, hand squared, metre squared, etc.). Pythagoras' theorem helps with squaring shapes.

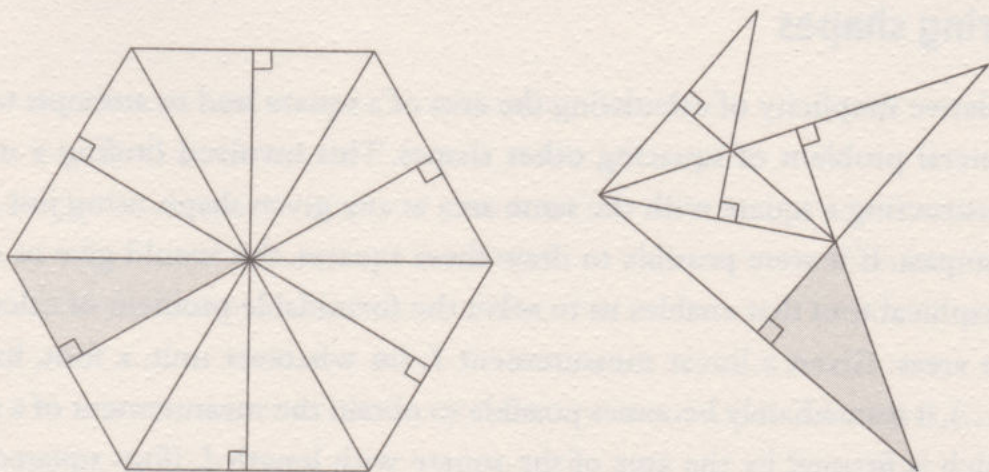
The first step is to learn how to square a triangle. Take a right-angled triangle with catheti b , c and hypotenuse a . As shown in the drawing, extend c by distance

b and mark the midpoint M of the segment $c + b$.



It is then possible to draw a circle with centre M and radius $(b+c)/2$. Then extending catheti b from A , to the point P on the arc of the circle. AP is the height of a right-angled triangle with hypotenuse $b + c$, by the theorem of height $PA = \sqrt{bc}$. Thus the square with side $\sqrt{bc}$ will have an area bc , and its midpoints will form a square (shaded in the figure) with half the area, or rather $1/2 bc$. Thus we have shown how to square any right-angled triangle.

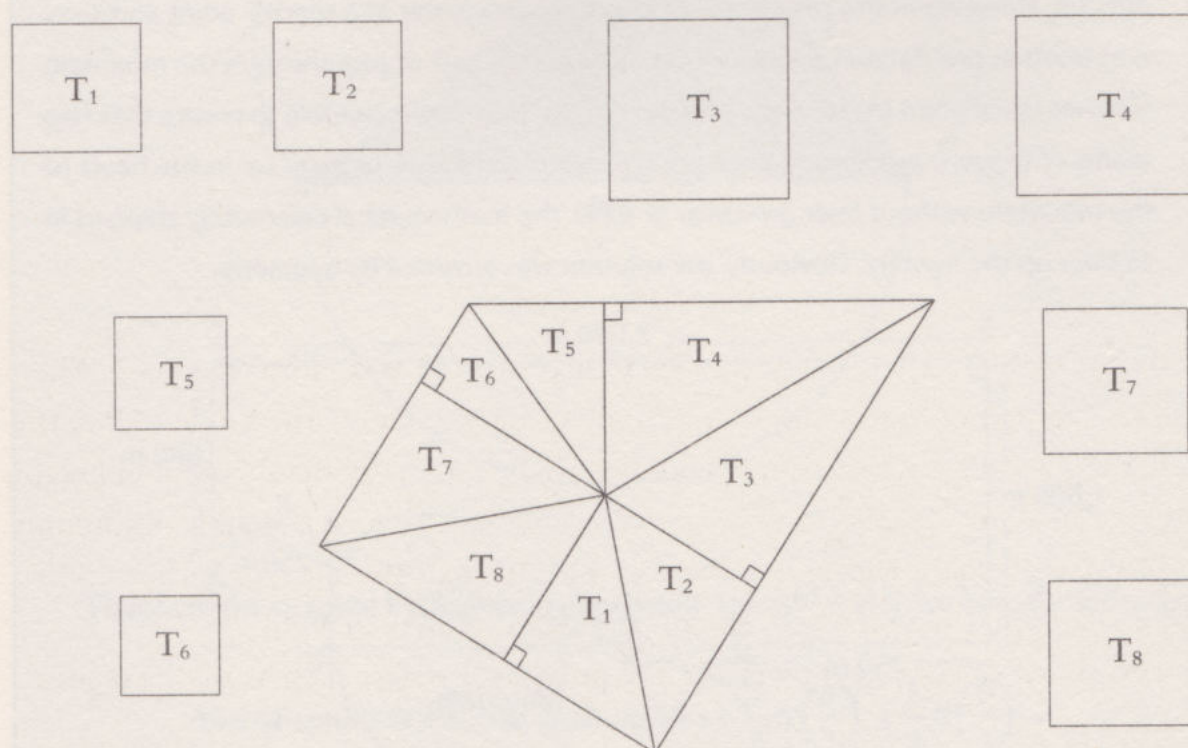
Now consider the problem for any polygonal shape, such as the hexagon.



As you can see above, any polygon can always be divided into triangles; and in turn, each of these triangles can always be divided into two right-angled triangles.

Thus it is always possible to calculate the area of the complete polygon based on the areas of its constituent right-angled triangles.

Thus decomposing a hexagon into right-angled triangles results in a series of squares that add up to the same area (see the figure below). It is now possible to use these to square the polygon. All the small squares are used to construct a single larger square, the area of which is the total of their areas. And thus an old friend surprises us once again: Pythagoras' theorem enters the scene. It is the key element in solving the problem of squaring shapes.

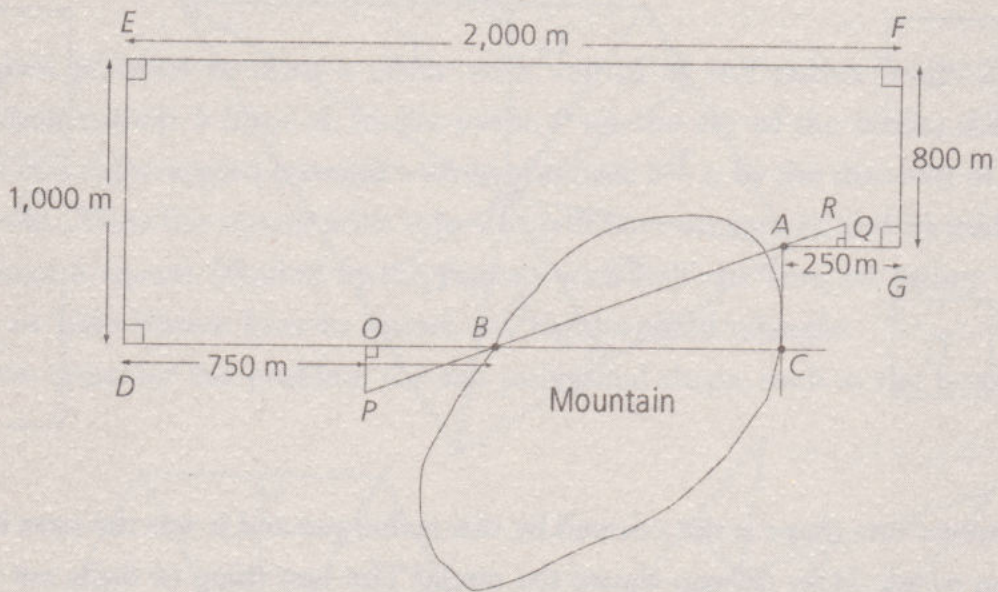


However one shape is not covered by this technique and it was the next logical question to ask: How do you square the circle? The best thing to do is not to try. Scholars have frittered away entire lifetimes – adding up to centuries of man hours – in the pursuit of this chimeric goal, this fusion of opposites. In the realm of physics, the transmutation of base metals into gold was an impossible dream, and its equivalent in mathematics is squaring the circle. The impossibility stems from the fact that the circle cannot be divided into triangles. It is possible to try to transform a circle into a polygon with a large number of extremely small sides and then square this, however this only gives an approximation of the square. It is just not possible to do this with a ruler and compass. In fact, the expression 'squaring the circle' is used to refer to an impossible task that is futile to pursue.

THE TUNNEL OF SAMOS

In 1882, a group of archaeologists discovered a 3,500-year-old tunnel on the Greek island of Samos. It had been built to carry water to the capital, Samos, through Mount Castro, and the Greek historian Herodotus referred to it as an incredible technological feat of the time. The tunnel is 1 kilometre long, 2 metres wide and 2 metres high.

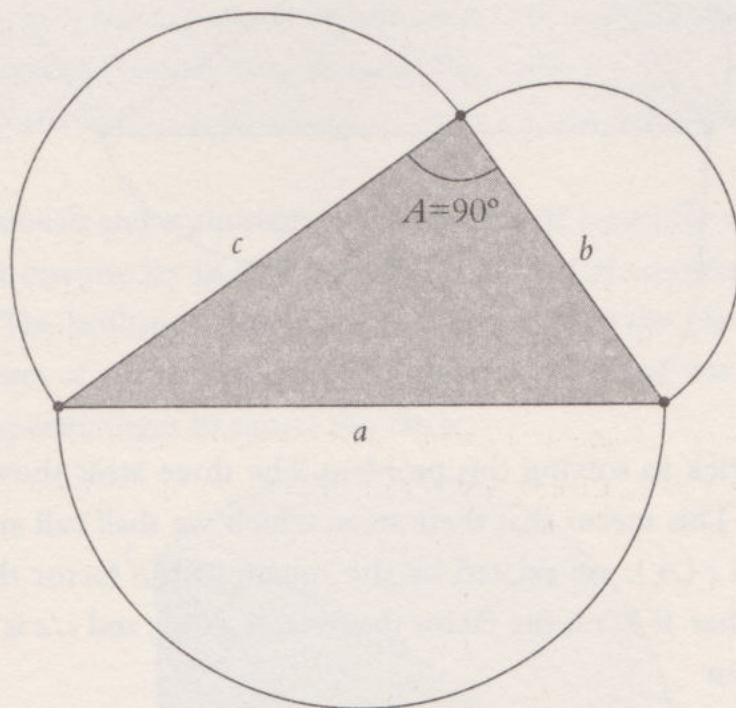
Its discovery caused amazement among engineers and mathematicians alike, since it highlighted an evident problem. Starting a tunnel anywhere and digging until finding something by chance or coming out somewhere else, not worrying about where, is not difficult. However in this case, the idea was to collect water at a specific point and carry it to another pre-defined point. The tunnel was required to pass through the mountain from an initial point (A) to reach a final point (B). How was it possible to ensure that two teams of diggers, working in opposite directions were able to meet up in the heart of the mountain without laser guidance or GPS? The mathematical community stepped in to clear up the mystery. Obviously, the solution was provided by geometry.



From B, travel distance BD (750 m in the example), turn 90° and travel DE (1,000 m), turn 90° and travel EF (2,000 m), turn again and we have FG (800 m), such that AG (250 m in the example) is perpendicular to FG. (Note that the idea is to determine perpendicular directions, taking suitable distances so as to be able to follow them on the outside of the mountain.) Here we have $BC = 1,000$ m and $AC = 200$ m. Pythagoras' theorem gives us a tunnel AB of 1,019.8 m. Moreover, if we draw the triangles BOP, which is similar to BAC (e.g. $BO = 50$ m, $OP = 10$ m), and ARQ with equal proportions, we know that the direction for digging is a line connecting PB and RA.

The sum of similar shapes

We have seen that Pythagoras' theorem solves the graphical sum of squares. The three squares on the sides of a triangle are three shapes with a proportional form. But what happens if these squares are replaced by other shapes, such as semicircles?

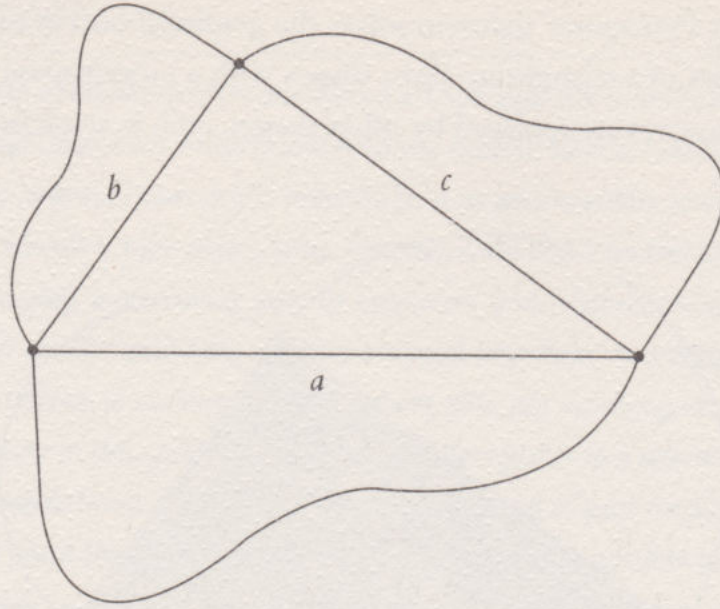


Thanks to the original Pythagorean expression ($a^2 = b^2 + c^2$), we have the following:

$$\begin{aligned} \text{Area of semicircle } b + \text{Area of semicircle } c &= \frac{1}{2} \pi \left(\frac{b}{2} \right)^2 + \frac{1}{2} \pi \left(\frac{c}{2} \right)^2 = \\ &= \frac{1}{2} \pi \frac{b^2}{4} + \frac{1}{2} \pi \frac{c^2}{4} = \frac{1}{2} \pi \left(\frac{b^2 + c^2}{4} \right) = \frac{1}{2} \pi \frac{a^2}{4} = \frac{1}{2} \pi \left(\frac{a}{2} \right)^2 \end{aligned}$$

or rather, the areas of the semicircles on the catheti add up to the area of the semicircle on the hypotenuse.

And what happens if the semicircles are replaced by much more complex curved shapes?



There is a trick to solving this problem. The three areas shown are actually the same shape. This means that their areas, which we shall call area a (A_a), area b (A_b) and area c (A_c), are related by the square of the factor that relates one length to the other. If b/a is the factor that relates a to b , and c/a is the factor that relates a to c , then

$$A_b + A_c = \left(\frac{b}{a}\right)^2 A_a + \left(\frac{c}{a}\right)^2 A_a = \frac{b^2}{a^2} A_a + \frac{c^2}{a^2} A_a = \left(\frac{b^2 + c^2}{a^2}\right) \cdot A_a = \frac{a^2}{a^2} A_a = A_a$$

Thus, the extension of Pythagoras' theorem allows any similar shapes to be added up graphically, regardless of how strange they look. Although the calculations depend on the classical proof of the case of squares, the theorem works across hundreds of other problems.

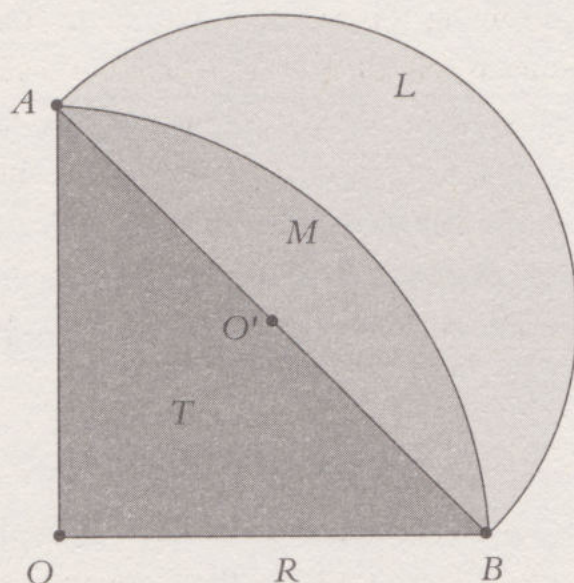
The lunes of Hippocrates

With the above proofs on record, the world of mathematics entered into a new dimension, in which all polygons could be squared using a ruler and compass. However foolish it may seem now, it was natural to suspect that it must also be possible to square a perfect circle by using either internal shapes or larger ones that surround it and using the simplest drawing tools. The long wait for indisputable proof that it was impossible lasted 2,000 years. In the meantime there were a number of centuries

HIPPOCRATES OF CHIOS

Hippocrates of Chios was one of the first Greek scholars to produce a detailed treatise on geometry (in around 440 B.C.). He is not to be confused, as is often the case, with the famous Greek doctor, Hippocrates of Cos (460-357 B.C.), who gives his name to the medical oath. Hippocrates used geometry to study how to duplicate the cube without the need for a ruler and compass, and proved the squaring of shapes, alongside many other results.

in which professionals and enthusiasts alike racked their brains for no return. Nevertheless, a great surprise lay in store for the mathematical community: the 'lunes' of Hippocrates. The brilliant Hippocrates of Chios showed the existence of shapes formed by the arcs of circles that could be squared. His proof was responsible for boosting the desperate urges to square the circle.



The figure above shows the lune or moon shape bounded by:

1. The quadrant of the circle with centre O and radius R .
2. The semicircle with radius $R\sqrt{2}/2$ and centre in the midpoint O' of the hypotenuse AB .

If T is the area of the right-angled triangle AOB (which has value $R^2/2$), L is the area of the lune and M is the area between the hypotenuse and lune, then:

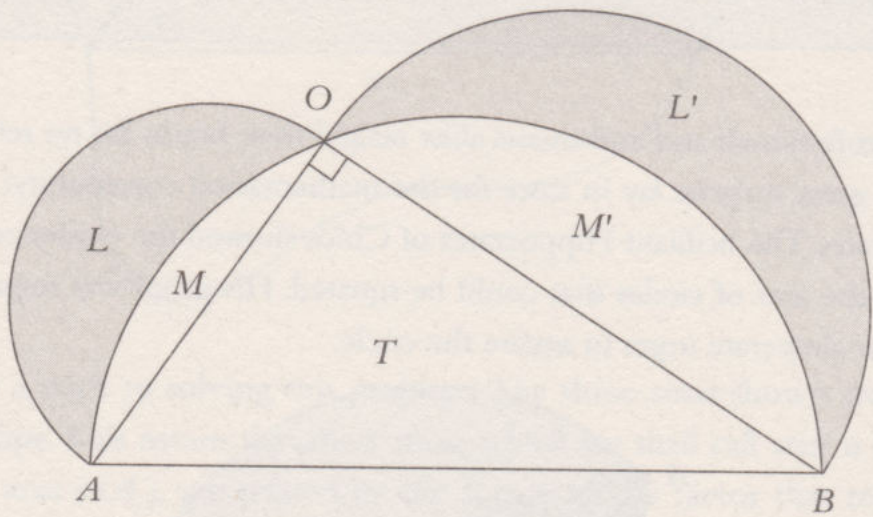
$$L + M = \frac{1}{2}\pi \left(R \frac{\sqrt{2}}{2} \right)^2 = \frac{\pi}{4} R^2,$$

which gives

$$L = \frac{\pi}{4} R^2 - M = T.$$

Hence the area of the lune is T ... and it is thus possible to square any lune.

But this was not Hippocrates' only lune. The most intriguing of his proofs can be seen in the following figure:



The semicircles on the catheti AO and OB have areas $L+M$ and $L'+M'$. In the previous section, we saw that the sum of the areas of these similar semicircles on the catheti is equal to the area of the semicircle on the hypotenuse, that is $T+M+M'$, or

$$L+M+L'+M'=T+M+M',$$

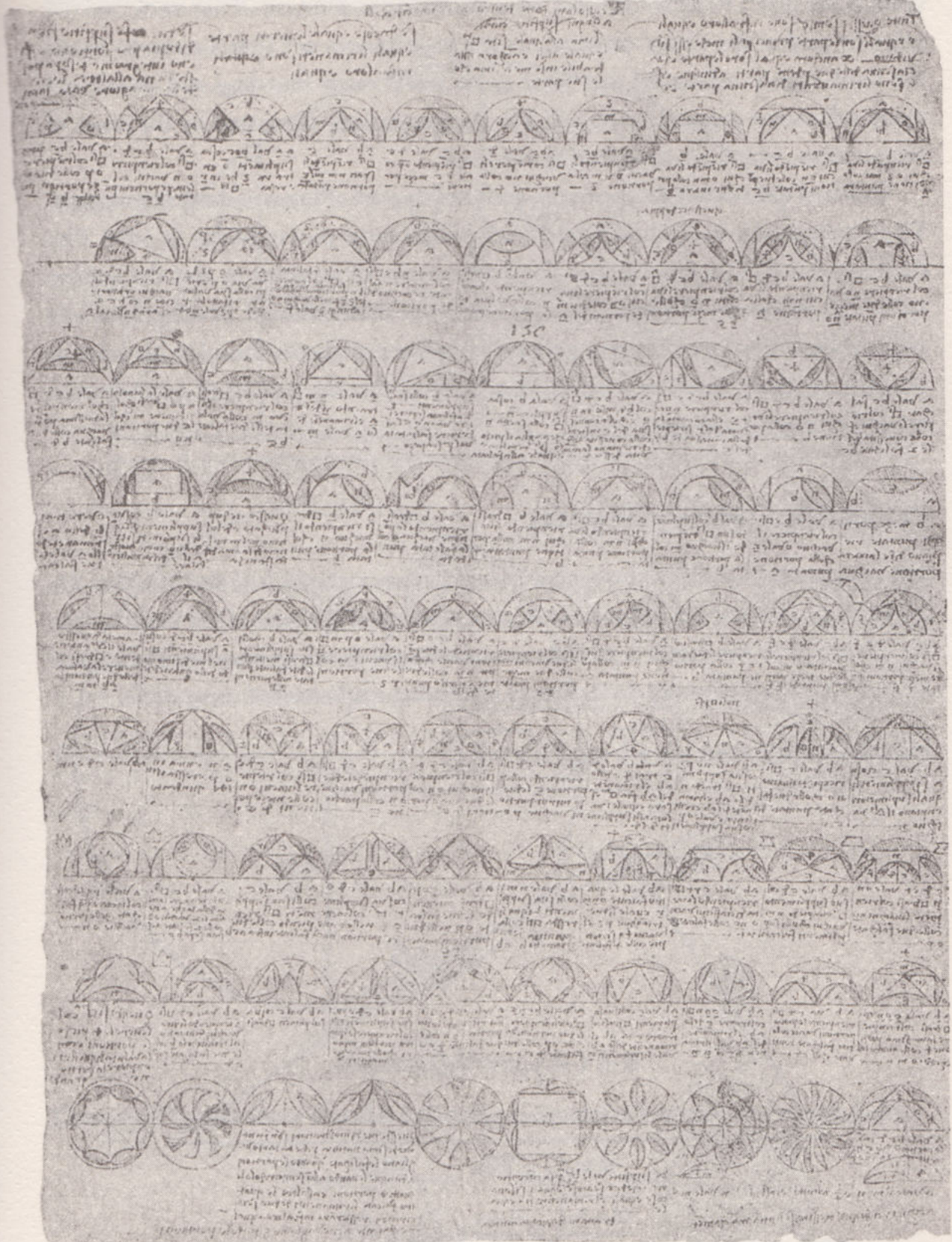
thus

$$L+L'=T$$

Hence the double lune has the same area as the right-angled triangle and can be squared.

Leonardo da Vinci and the lunes

On 10th November 1494, Luca Pacioli published his great work *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*. It sold well among his contemporaries, who were eager to buy a copy, although it is not known how many read it, let alone understood it. One of the buyers was a fellow countryman of Pacioli, who acquired a copy for 129 salaries and passionately devoured its pages. The eager buyer was



A page from da Vinci's Codex Atlanticus, identified as f 455r, with drawings and notes referring to the lunes of Hippocrates.

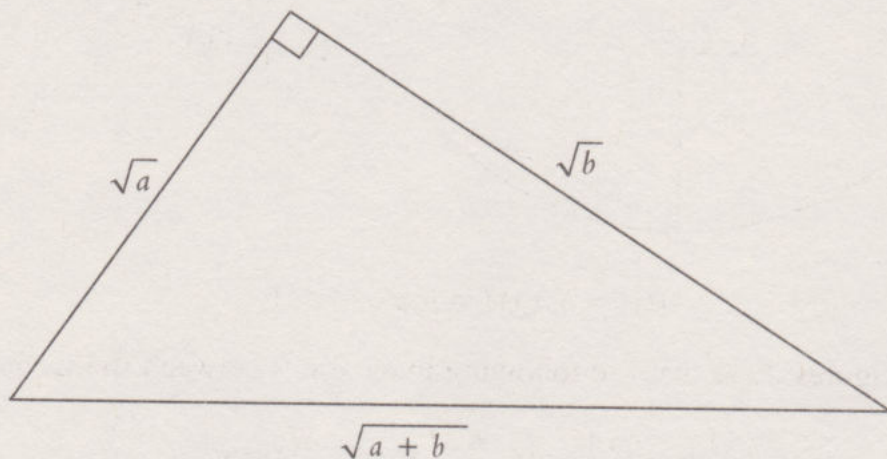
none other than Leonardo da Vinci, who noted the acquisition of the book and the price he paid in the *Codex Atlanticus*, on the page known as f 104r. Leonardo found the book highly influential and went on to make many drawings and sketch out many reflections inspired by Pacioli. He was particularly fascinated by the lunes of Hippocrates, which are described in the book.

Da Vinci discovered an impressive series of squares of lunes and other polygons. These surprised even Pacioli himself, whom the artist met in Milan in 1496. The drawings led da Vinci to explore the mathematics of proportion and the golden ratio. The influence of these concepts and their mathematical development not only made da Vinci a prominent geometrician but can also be found in his work as an artist. Nobody has shown better than da Vinci that the perceived gulf between art and science is a fiction.

Inequalities with Pythagoras

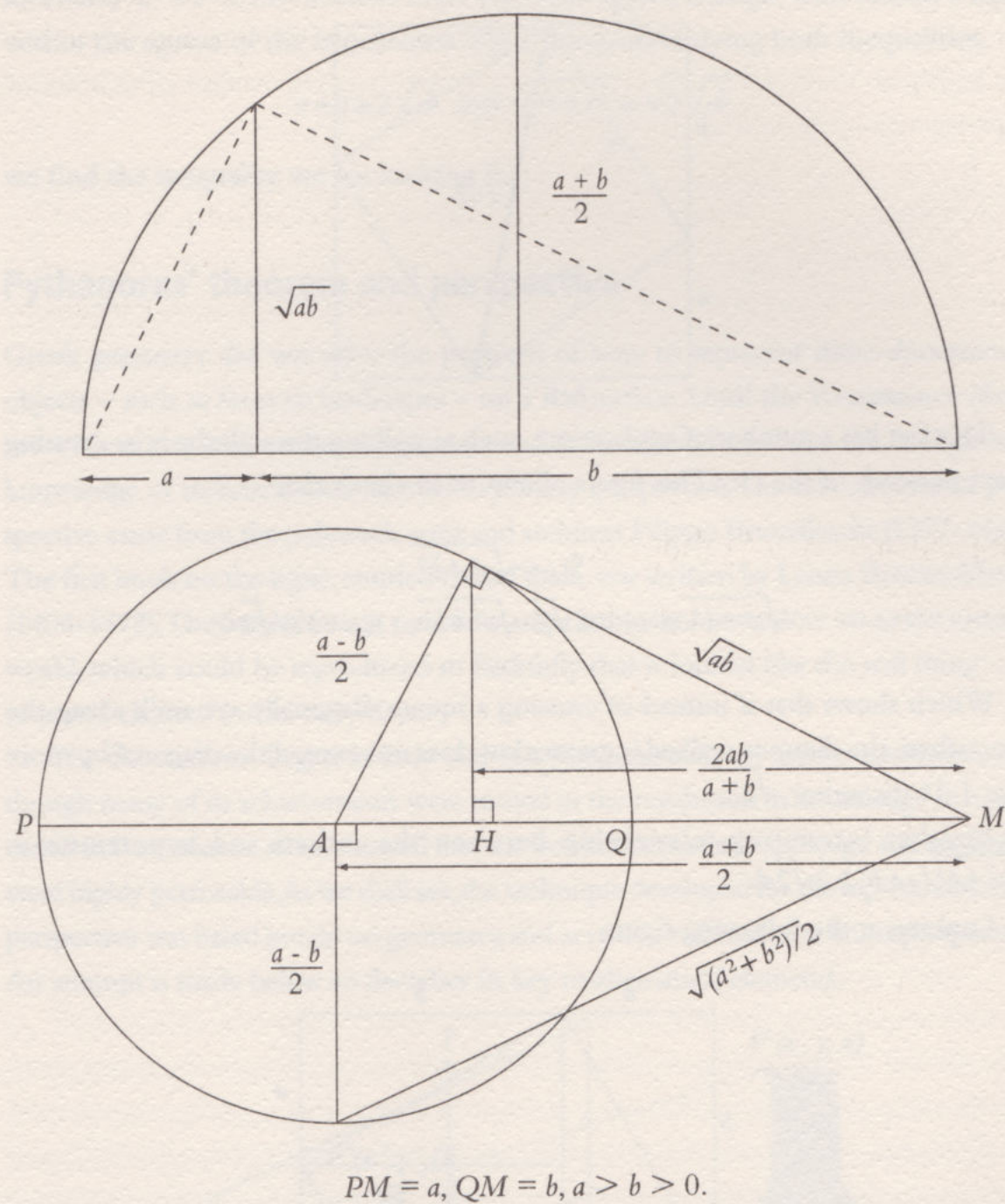
It is obvious that for any right-angled triangle, either cathetus is always less than the hypotenuse and that the hypotenuse is always less than the sum of the catheti. Taking advantage of this fact, it is possible to make simple drawings based on Pythagoras' theorem, which lead to the deduction of interesting mathematical inequalities.

Inequality between $\sqrt{a+b}$ and $\sqrt{a} + \sqrt{b}$



Thus: $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$.

Inequalities between arithmetic means and the geometric mean

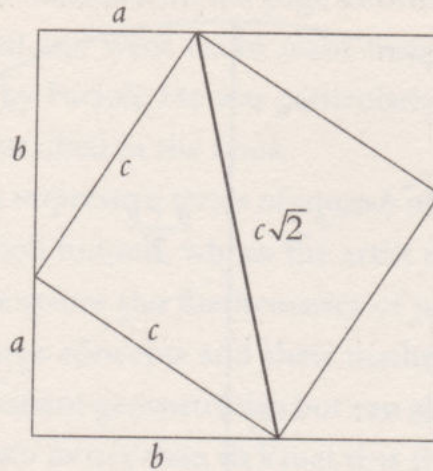


These two figures show that the following inequalities between the means hold:

$$M_{\text{Harmonic}} \leq M_{\text{Geometric}} \leq M_{\text{Arithmetic}} \leq M_{\text{Square root}}$$
$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}.$$

That is to say, the harmonic mean is less than the geometric mean, which is in turn less than the arithmetic mean, which is in turn less than the square root.

Inequalities between the hypotenuse and catheti



This fact has a number of applications, such as walking through the criss-crossing street network of the city. This figure allows us to check that

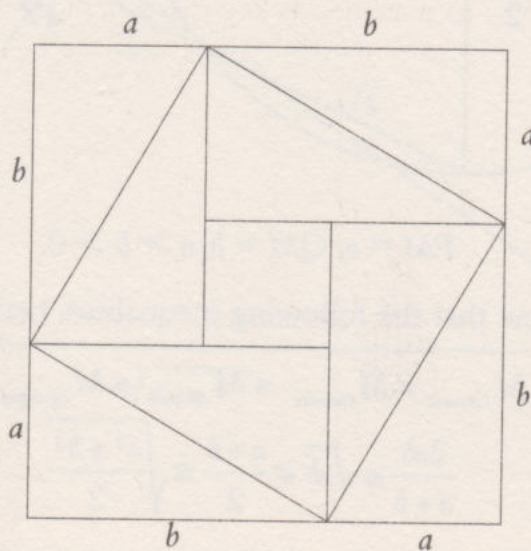
$$\text{Si } c = \sqrt{a^2 + b^2}$$

$$\sqrt{a^2 + b^2} \leq a + b \leq \sqrt{2} \cdot \sqrt{a^2 + b^2}; \quad c \leq a + b \leq \sqrt{2} \cdot c.$$

Which shows that if instead of crossing a square diagonally, we walk along the two catheti, the distance walked is greater but does not exceed the diagonal by more than 1.41 (or rather, $\sqrt{2}$).

Another interesting relationship between the catheti and hypotenuse is:
 $\sqrt{a^2 + b^2} (a + b) \geq 2\sqrt{2}ab$.

Looking at the following figure



you can see that $(a+b)^2 \geq 4ab$.

This is because the square of length $a + b$ contains rectangle with sides a and b ; moreover $a^2 + b^2 \geq 2ab$ because there are right-angled triangle with catheti a and b within the square of the hypotenuse. Thus, upon multiplying both inequalities,

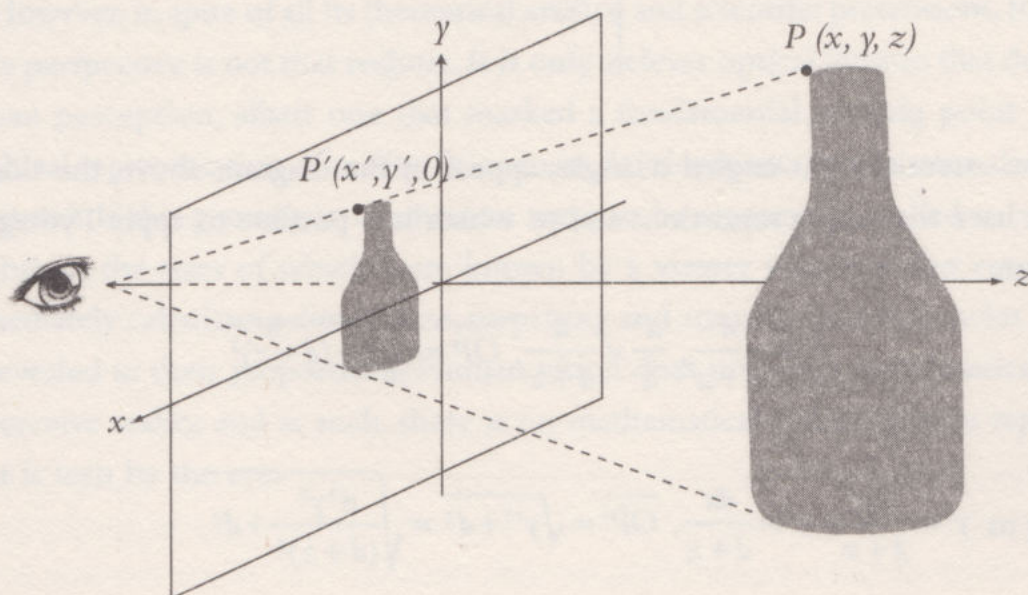
$$a + b \geq 2\sqrt{ab} \text{ and } \sqrt{a^2 + b^2} \geq \sqrt{2}\sqrt{ab},$$

we find the inequality we are looking for.

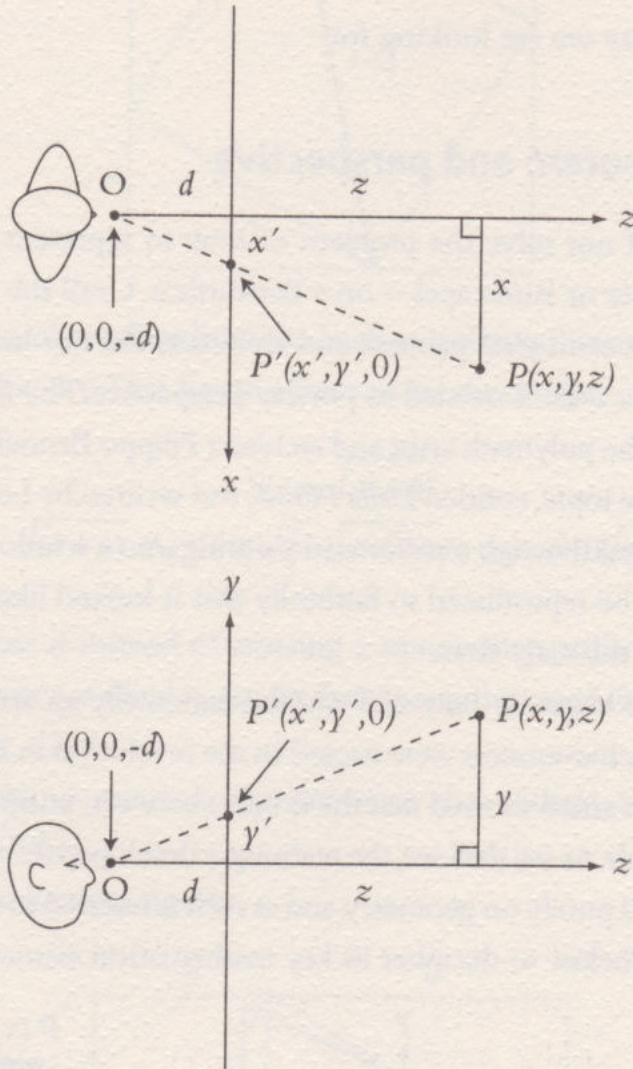
Pythagoras' theorem and perspective

Greek geometry did not solve the problem of how to represent three-dimensional objects – such as faces or landscapes – on a flat surface. Until the Renaissance (from the 14th to the 16th centuries) painters and architects did not have the background knowledge or technical skills needed to portray perspective. The first advance in perspective came from the polymath artist and architect Filippo Brunelleschi (1377–1446). The first book on the topic, entitled *Della Pittura*, was written by Leone Battista Alberti (1404–1472). The breakthrough transformed painting into a window on to the outside world, which could be reproduced so faithfully that it looked like the real thing – or so it was claimed in the early days.

The Renaissance began in Italy and was fundamentally an artistic movement, although many of its achievements were rooted in the revolution in knowledge that preceded it. Renaissance artists showed that the borders between knowledge and aesthetics were highly permeable. As we shall see, the technique developed for the representation of perspective was based purely on geometry, and as such is referred to as linear perspective. An attempt is made below to decipher its key mathematical elements.



As can be seen in the previous figure, the key to perspective is to consider four elements: the object, the painting, the eye of the painter and a virtual 'pyramid of rays' running from the eye of the observer to the real object. A representation of the object in true perspective – the painting – is a section or slice of this virtual pyramid – the projection.

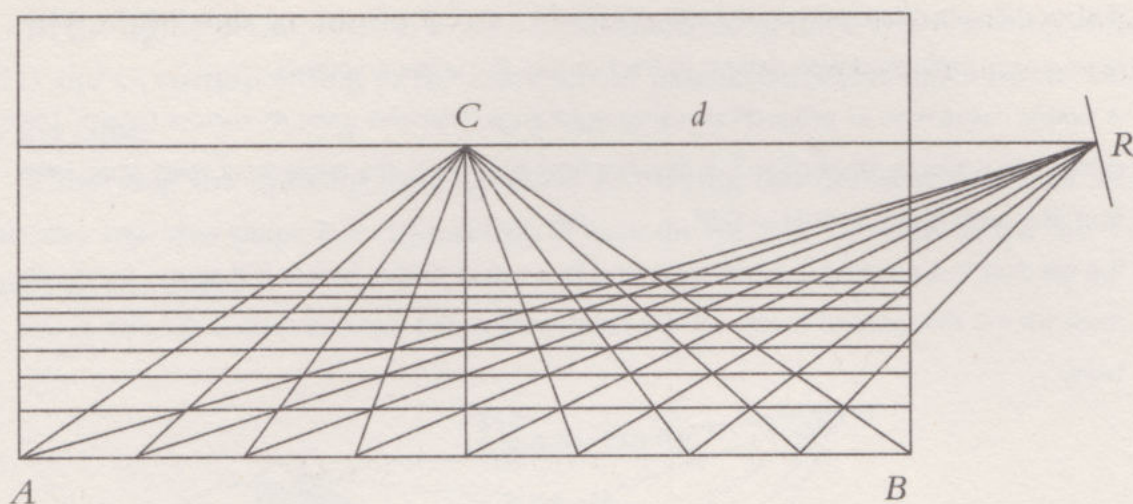


Two essential right-angled triangles appear in the diagram above, the sides of which have the same proportions and to which it is possible to apply Pythagoras' theorem:

$$\frac{y'}{y} = \frac{d}{d+z}, \quad \frac{x'}{x} = \frac{d}{d+z}, \quad \overline{OP} = \sqrt{y^2 + (d+z)^2}$$

giving us $y' = \frac{dy}{d+z}, \quad x' = \frac{dx}{d+z}, \quad \overline{OP'} = \sqrt{y'^2 + d^2} = \sqrt{\frac{d^2 y^2}{(d+z)^2} + d^2}.$

Of course, a painter does not normally work on a transparent pane of glass, but on an opaque canvas, and so artists had to become geometers. Consider, for example, Alberti's *Costruzione Legittima*.



Alberti tries to represent a pavement in the form of a grid. To do so, he fixes a central vanishing point C , divides AB into equal parts and joins them all to C . He then locates R at a distance d from point C . This distance d is equal to the one separating the artist from the painting. He then joins R with the divisions of AB . This gives the diagonals of the paving stones that can now be used to represent them.

Someone viewing the painting can find the ideal distance to observe the mosaic of paving stones. It depends on reproducing Alberti's path in the opposite direction. All that is required is a smaller copy of the painting that allows us to draw beyond its borders in order to trace the diagonals. These converge outside the painting at point R , and this makes it possible to calculate the distance $d = \overline{CR}$ to scale.

However, in spite of all its theoretical artifice and scientific pretensions, Renaissance perspective is not that realistic. It is only a clever optical illusion that deceives human perception, albeit one that marked a fundamental turning point in the graphical representation of reality. Prior to its development, images were distinctly unnatural. If this geometric mechanism were to be rigorously applied to a series of objects, the sizes of which were known by a viewer who was also capable of immediately calculating dimensions, surprising and amusing inconsistencies would be revealed in their proportions. Human vision does not use a mathematical trick to perceive reality, and as such, there is no mathematical system able to represent what is seen by the eye.

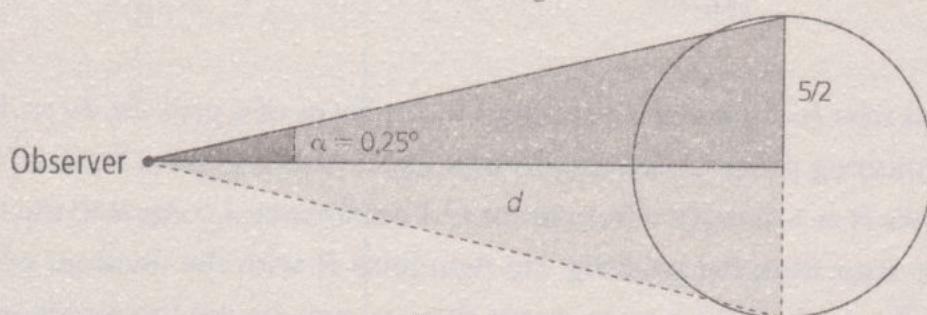
REPRESENTING THE MOON

In painting, theatre and even cinema, the moon often appears in night scenes, where its size and location are often represented erroneously. As a general rule, we can show that the lower the moon is positioned with respect to the horizon, the larger it appears.

A simple calculation of proportions using right-angle triangles gives its correct height. The calculation is simple, since given our distance from the moon, the angle it occupies when seen from the earth is tiny, as little as 0.5° .

If a window in a painting is 20 cm wide and four moon shapes fit into this space, the moon must have a diameter of 5 cm... If the painter views the painting from a distance d , we have,

$$\tan (0.5^\circ / 2) = \frac{5/2}{d},$$



or rather, $d = 2.5 / \tan 0.25^\circ \approx 581.4$ cm. That shows that this is the work a giant painter who can reach forward with his brush from 5.81 m in front of the painting.

From where do I view a painting?

Where did Piero della Francesca and Leonardo da Vinci want viewers to stand to admire their work?

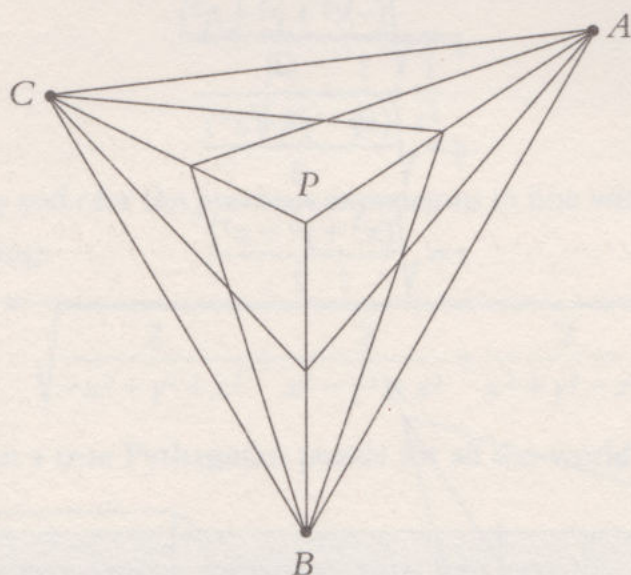
The strict principles of linear perspective described by the Florentine architect, artist and theoretician Filippo Brunelleschi establish that the pictures as a representation of reality must offer the viewer's eye the same vision that the artist wished to show in the creation of their work. Any set of parallel lines seen in reality are represented in the painting as converging on a vanishing point or a point at infinity.

This vanishing point has to be the point of the drawing that intersects with the theoretical line 'passing through the eye of the artist' which runs in parallel to the family of the previous straight lines. As such, the ideal view of the painting occurs

when the viewer stands at the point directly in front of the painting where the artist was standing when he created it. But is it possible to discover where this point is? The mathematician Chris Zeeman shows us how to do so.

In the figure below, there is a cube represented using the three vanishing points A , B and C , corresponding to the three sets of parallel lines that make up the edge of the cube.

Observing the drawing from up close and trying out different points of view, the idea that this shape is a cube is lost. Where do we have to stand to see the true shape of the cube?



By definition all straight lines that form a set of parallel lines in three dimensional reality pass through each vanishing point of the painting. To 'see' that AP and PB make up a right angle (and thus form part of a cube) it is necessary to make use of a little trick. Imagine that there is a sphere with diameter AB in front of the painting. The viewer's eye should be at a point on the surface of the hypothetical sphere. This has to be the case because all the lines within a 90° field of vision can be seen from a spherical surface. However, in order to see the cube, it is necessary to see the perpendiculars AP and PC , and the same with BP and PC . Thus it is best to have the eye at the intersection of three hypothetical hemispherical surfaces in front of the painting which have diameters AB , BC and AC , respectively. The first two hemispherical surfaces in front of the painting will be cut in a semicircle, and this will intersect the third hemispherical surface at a single, unique point.

A few calculations allow us to deduce that the distance d at which the viewer's eye must be located in order to view the cube properly in all its glory. Let $AB = x, BC = y, CA = z$, and from the view point O the distances to A, B, C are q, r and p , respectively.

In no less than three applications of Pythagoras' theorem, we get:

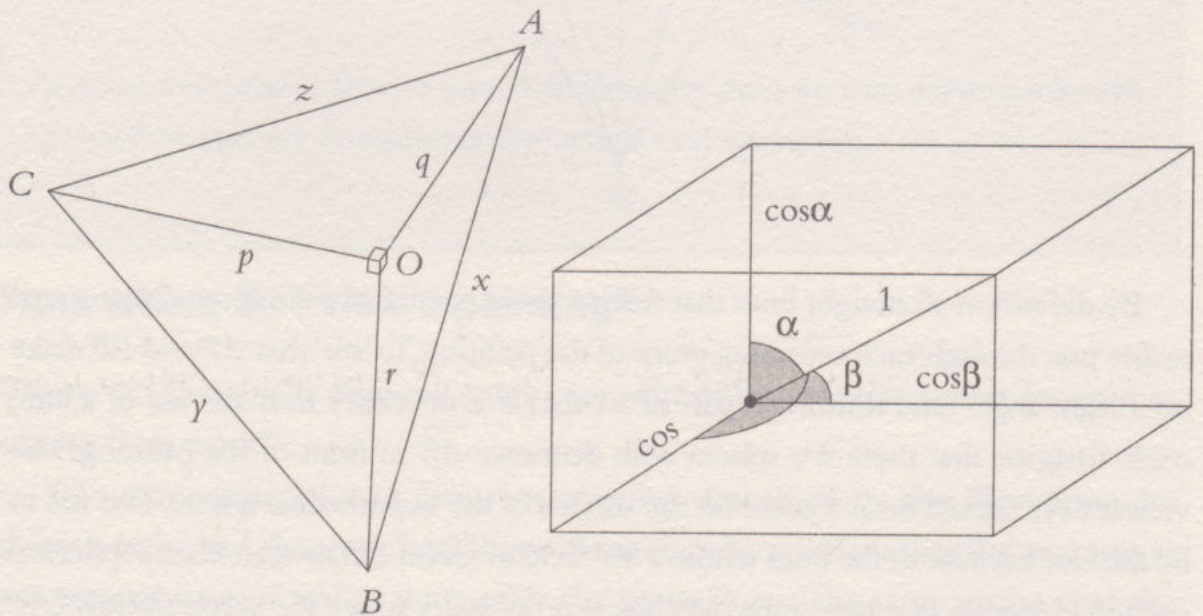
$$x^2 = q^2 + r^2, y^2 = p^2 + r^2, z^2 = p^2 + q^2,$$

and solving this system of equations (x, y, z can be measured in the painting), we obtain the values of p, q and r :

$$p = \sqrt{\frac{-x^2 + y^2 + z^2}{2}}$$

$$q = \sqrt{\frac{x^2 - y^2 + z^2}{2}}$$

$$r = \sqrt{\frac{x^2 + y^2 - z^2}{2}}$$



In the figure on the right, it is possible to see that if the diagonal of a box has length 1 then the angles of the diagonal with the edges are α, β, γ . By Pythagoras' theorem

$$\cos \alpha = \frac{\text{adjacent cathetus}}{\text{hypotenuse}} = \frac{\text{adjacent cathetus}}{1},$$

and thus $\cos \alpha = \text{adjacent cathetus}$ and coincides with the axis of the coordinates.

$$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$$

thus,

$$1 = \left(\frac{d}{p}\right)^2 + \left(\frac{d}{q}\right)^2 + \left(\frac{d}{r}\right)^2,$$

$$1 = \frac{d^2}{p^2} + \frac{d^2}{q^2} + \frac{d^2}{r^2} = \frac{d^2 q^2 r^2 + d^2 p^2 r^2 + d^2 p^2 q^2}{p^2 q^2 r^2} = d^2 \left(\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} \right) = 1$$

giving d as

$$d = \frac{1}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}}}.$$

Substituting p, q and r for the previous expressions in line with the measurements x, y, z of the painting:

$$d = \frac{1}{\sqrt{\frac{2}{-x^2 + y^2 + z^2} + \frac{2}{x^2 - y^2 + z^2} + \frac{2}{x^2 + y^2 - z^2}}}.$$

Which results in a true Pythagorean puzzle for all the world's galleries.

PYTHAGORAS' THEOREM, RIBBONS AND STAKES

Looking back at the origins of architecture, the first step in the construction of any building is to mark out the land beforehand. This means that the problem of precisely defining a section of land and the solution of marking it with a rectangle is several millennia old. At first sight, the operation seems trivial. All that is required are four stakes to put into the ground, strings and a measuring tape. But the question of precision still hangs in the air. How is it possible to verify that a true rectangle has been created and not just any old quadrilateral? Just like in medicine, architecture can only tolerate a small degree of imprecision.

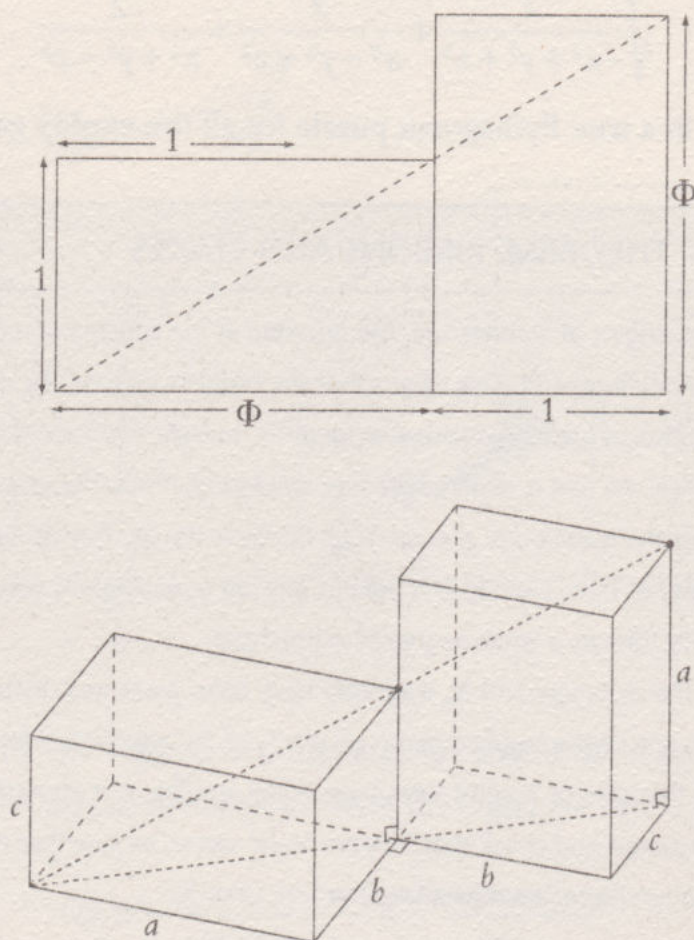
Once the sides of the rectangle and its diagonals have been measured, Pythagoras' theorem makes it possible to ensure the required perpendicularity of the adjoining sides of the rectangle. But simpler still, if the pairs of parallel sides have equal lengths, it is enough to measure the diagonals of the rectangle and check that they are equal and so be sure that the shape marked out a rectangle and not just any old parallelogram.

Van der Laan's plastic number

In the 20th century, the Dutch architect and Benedictine monk, Hans van der Laan (1904-1991) discovered a new candidate to swell the list of numbers with surprising properties that have always fascinated the mathematical community. It was named the mystical or plastic constant, not in reference to the era of plastic in which it was discovered, but in the noble sense of aesthetic plasticity. The number P is an especially interesting case of cubic roots. It arises in the positive solution to the cubic equation $x^3 = x + 1$, or rather

$$P^3 = 1 + P,$$

where $P = 1.32\dots$ Just as the golden ratio Φ gives us $\Phi^2 = \Phi + 1$, by proving, $P^3 = P + 1$, the plastic constant represents what the golden number is in the two-dimensional plane for many situations in three-dimensional space. See the following figure where the box with edges $c < b < a$ has been cloned and repositioned:



The property of the rectangles characterises Φ . As shown in a previous figure, if we extend the main diagonal of the horizontal box in space so that it passes through the upper corner of the vertical box, we can only do this if $b=Pc$, $a=P^2c$ with c being an arbitrary value and P the plastic constant. In order to prove this result, Pythagoras' theorem can be applied to the right-angled triangles associated with the diagonals in the figure opposite.

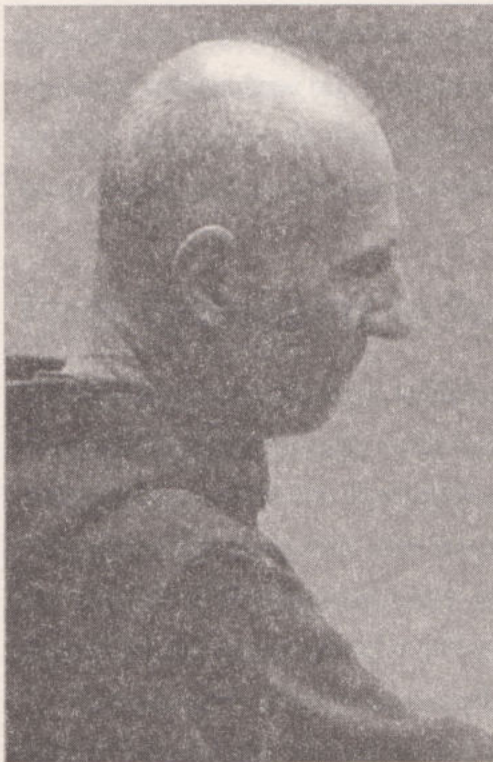
Father Van der Laan studied the proportions of Roman churches and discovered that many corresponded to the proportions of the terms in the following series:

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \dots$$

The series begins with three ones and each term is obtained by adding the numbers that come 2 and 3 positions before it ($2 + 3 = 5$; $4 + 5 = 9$), resulting in the following quotients

$$\frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{2}{1}, \frac{2}{2}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{7}{5}, \frac{9}{7}, \dots$$

These quotients tend towards the plastic constant P .



The work of Father Hans van der Laan has been taught and studied in many universities throughout the world.

Among the first creations to materialise from his theoretical proposals were the upper church, atrium and crypt (shown above) of the abbey in Vaals, in the south of Holland.

Chapter 6

Beyond the Theorem of Pythagoras

Time is the soul of this world
Pythagorean principal

The final straight of our Pythagorean journey is drawing near, but the hard work is not over just yet. Up until now we have looked at the importance of Pythagoras' theorem from various perspectives but always in its traditional form, as the relationship between the areas of squares associated with a right-angled triangle. However, we have only touched the surface of its great depths. A vast territory now stretches out before the traveller, an unknown and often thrilling land. We are going to take a few small steps forwards, but that is enough for us to enter a new dimension. We will see results that extend the theorem, visit new conditions where Pythagorean relationships form, and discover completely unexpected appearances of the theorem. To do this, the travellers will have to use (or even abuse) their imagination to visualise things that are beyond the boundaries of space and time. They will travel to that non-place to discover how one of humanity's oldest proofs still poses the most novel and unforeseen challenges.

From Pythagoras to Fermat and on to Wiles

The Pythagorean triangle, which has been a well-known shape since ancient times is the right-angled triangle with sides of 3 and 4 and a hypotenuse of 5 ($5^2 = 3^2 + 4^2$). After its discovery the Pythagoreans asked themselves: are there other right-angled triangles with sides that are whole numbers? In order to find them, according to the theory of Pythagoras, they had to search for non-zero positive integers x , y , z linked by the equation $x^2 + y^2 = z^2$. (1)

Pythagorean triples are combinations of three numbers that fit this equation. The Pythagoreans toiled in the search for triples, a special form of mathematical

relationship. Did the triples become more and more scarce as the numbers got larger? The Pythagoreans invented a methodical way to produce the maximum possible number of them. On the way, they demonstrated that there are an infinite number of Pythagorean triples.

In book X of Euclid's *Elements*, the formula that allows all the integer solutions to (1) to be found already appears. Taking any two positive integers m and n where $m > n$, one even and the other odd, use

$$\begin{aligned}x &= 2mn \\y &= m^2 - n^2 \\z &= m^2 + n^2.\end{aligned}\tag{2}$$

And we get all the Pythagorean triples, or at least as many as we have time to reveal since there are an infinite number. The first few values can be seen in the following table:

m	n	x	y	z
2	1	4	3	5
3	2	12	5	13
4	1	8	15	17
4	3	24	7	25
5	2	20	21	29
5	4	40	9	41
6	1	12	35	37
6	5	60	11	61

Notice that if x, y, z is a Pythagorean triple, then ax, ay, az , where a is an integer, also is. It can be said that from any triple an infinite number of others can be deduced. Perhaps the most endearing one, though, is the good old 3, 4, 5, the only one with arithmetic progression (3, 3 + 1, 3 + 2), which produces the only triangle with integer sides whose perimeter, which is 12, is twice its area, which is 6.

Greek mathematician Diophantus of Alexandria wrote three great works, one of which was highly influential with its title providing the name for an entire discipline of mathematics. The book was entitled 'The science of numbers', which was just one word in Greek made up of 'arithmos' (number) and 'techne' (science), in other

A CURIOUS TRIPLE

666 or the Number of the Beast is and always has been an enigmatic number that has aroused great passion (of all types) among fans of numerology. Its strange properties are evident in the Roman numeral system, where 666 is written as DCLXVI. As we can see, all the value symbols below M (1,000) are used just once in descending order. In strict mathematical terms, other relationships are also interesting: for example, 666 is the sum of the squares of the first seven prime numbers (2, 3, 5, 7, 11, 13, 17). 666 enthusiasts also have at their disposal a particularly interesting Pythagorean triple: $x = 693$, $y = 1,924$ and $z = 2,045$. The area of the triangle determined by this triple is... 666,666.

words 'arithmetike'. Diophantus' *Arithmetic* spurred on the development of number theory for centuries. In his work, Diophantus solved 130 problems and formulated the expression for solving Pythagorean triples (2) seen in Euclid's *Elements* but used different arguments. Pythagoras' theorem often seems to be at the centre of his problems. For example, problem 1 of Book VI contains the Pythagorean triple 40, 96 and 104, where the hypotenuse minus either of the other sides is a cubic number, as can be seen in the following calculation:

$$104 - 96 = 8 = 2^3,$$

$$104 - 40 = 64 = 4^3.$$

The influence of Diophantus' *Arithmetic* is not just in its content, but also through a twist of fate scribbled in the margins of its pages. It was in the margin of a printed copy of the classic *Arithmetic* where the multi-talented Pierre de Fermat (1601-1665) in around 1637, made the following teasing statement:

"It is impossible to divide a cube into two cubes, or a fourth power into two fourth powers, or generally any power beyond the square into like powers; of this I have found a remarkable proof. But this margin is too narrow to contain it".

A bold statement, yet the margin did not have enough space for it! The enigma that became to be known as Fermat's Last Theorem had the mathematical community baffled for no less than 300 years, as hundreds of mathematicians worked to find Fermat's so-called remarkable proof. Such is the difficulty posed by the problem

that it was decided that Fermat must have made a mistake. However, in 1995, Briton Andrew Wiles came up with a great solution.

It is not possible to express Wiles' demonstration in just a few lines, because it is a very high level of mathematics. Accept for the moment that for the values x , y , z , all non-zero positive integers, and $n \geq 3$, the equation $x^n + y^n = z^n$ is not fulfilled. After the enormous number of Pythagorean triples that can be found where $n = 2$, there is not a single one where it is greater than 2.

The impossibility of $n = 3$ had already been solved by Leonard Euler. Since then the most brilliant minds in the world of algebra and the theory of numbers dedicated themselves to producing results for the problem. Wiles presented his

THE AGE OF DIOPHANTUS

As with Pythagoras, no one knows much for certain about the life of Diophantus. There is speculation about the date of his birth and death, when really we cannot even be certain about which century he lived in. However, we do know, with the utmost mathematical certainty the age at which he died. His epitaph, written in the form of a problem, has been preserved in Greek literature as a popular riddle:

"Passerby, this is the tomb of Diophantus: it is he who with this surprising distribution tells you how many years he lived. Diophantus' youth lasted one sixth of his life. He grew a beard after one twelfth more. After one seventh more of his life, he married. 5 years later, he and his wife had a son who, once he reached half the age of his father, suffered an unfortunate death. His father had to survive him, crying, for four years. From this, deduce his age."

Let's summarise, removing the verbiage to clarify the premises: Diophantus spent one sixth of his life as a child, one twelfth as an adolescent and one seventh as an adult. Five years after his wedding a son was born who died four years before his father, when he was half the age of what would be the father's final age. At what age did Diophantus die?

Finding the solution is easy and good fun.

However, those who do not have the time or patience, could carry out the calculation simply with the following formula based on the premises.

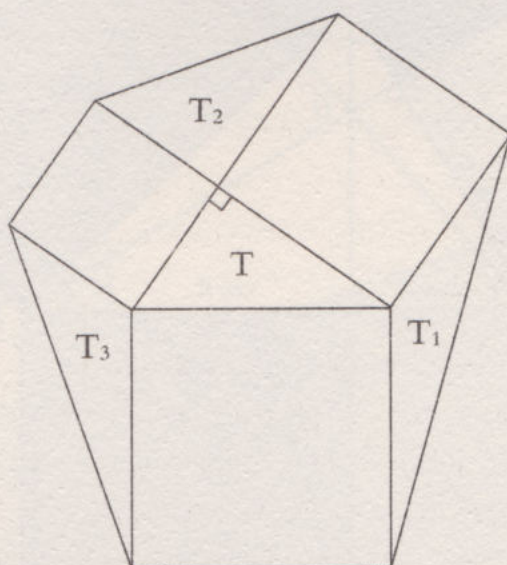
$$\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x.$$

first demonstration in 1993, but it turned out to have an error in it. He managed to correct it with the help of Richard Taylor and rose above the swirl of results to completely resolve the problem two years later.

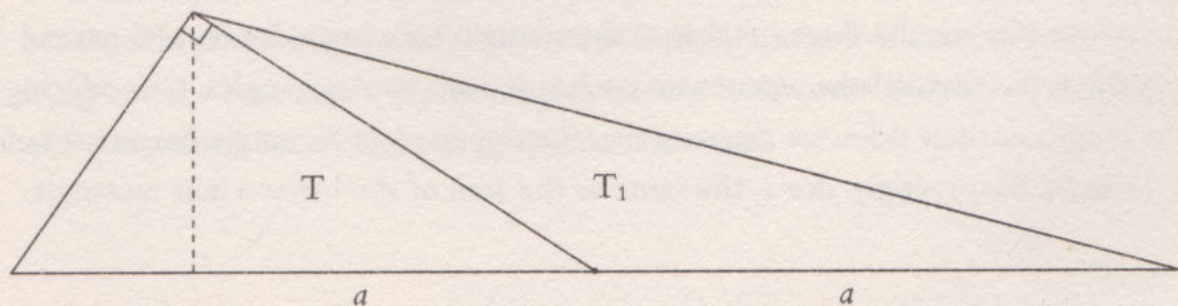
Pythagorean ratios in other polygons

The Greek Pythagorean ratio is linked to a very specific figure – the right-angled triangle. What happens to the squares on the sides of any old triangle? What happens to the squares of a parallelogram's four lengths?

Completing the Pythagorean figure

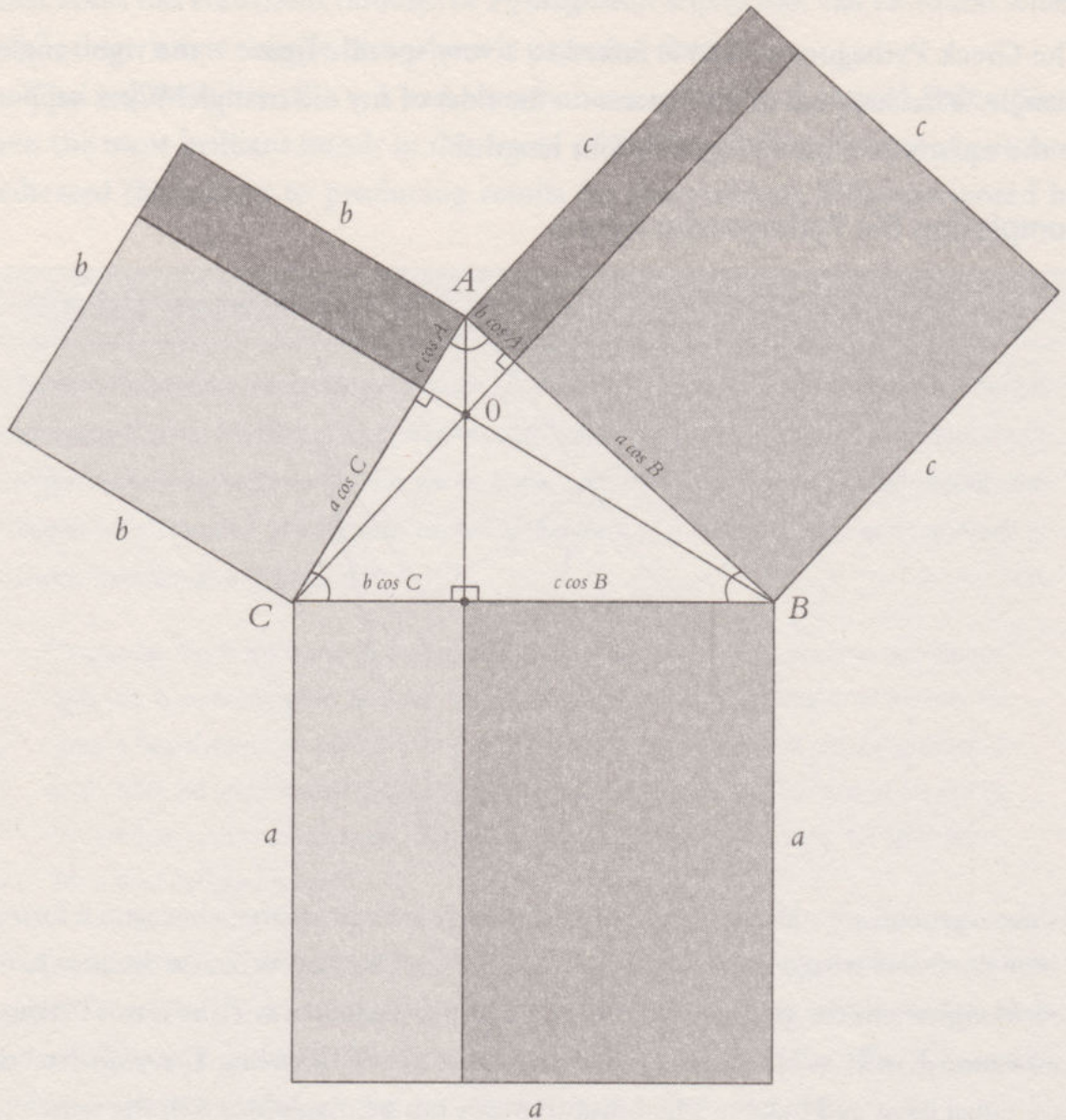


If three segments are added to a right-angled triangle with its squares, a hexagon is formed where three new triangles with areas T_1 , T_2 and T_3 are formed. What are the areas of the new triangles? The three are equal and have exactly the same area as T , the classic Pythagorean figure: $T_1 = T_2 = T_3 = T$. The figure shows that $T_1 = T$. Rotating T_1 reveals that both have equal bases and heights. The other triangles can be manipulated in the same way.



The cosine theorem

If ABC is any triangle (not necessarily right angled), it still makes sense to present the three squares on the sides and ask what the relationship between their areas is. For the utmost clarity, imagine the case of an acute triangle ($A < 90^\circ$). The following image leads us to the solution:



As we can see, the three heights of the triangle have been drawn, and extending them has divided the squares on each side into two rectangles. Considering the length of their sides, we discover that the upper-right rectangle has an area of $c \cdot (a \cos B)$. Surprisingly, this is the same as the area of the lower-right rectangle.

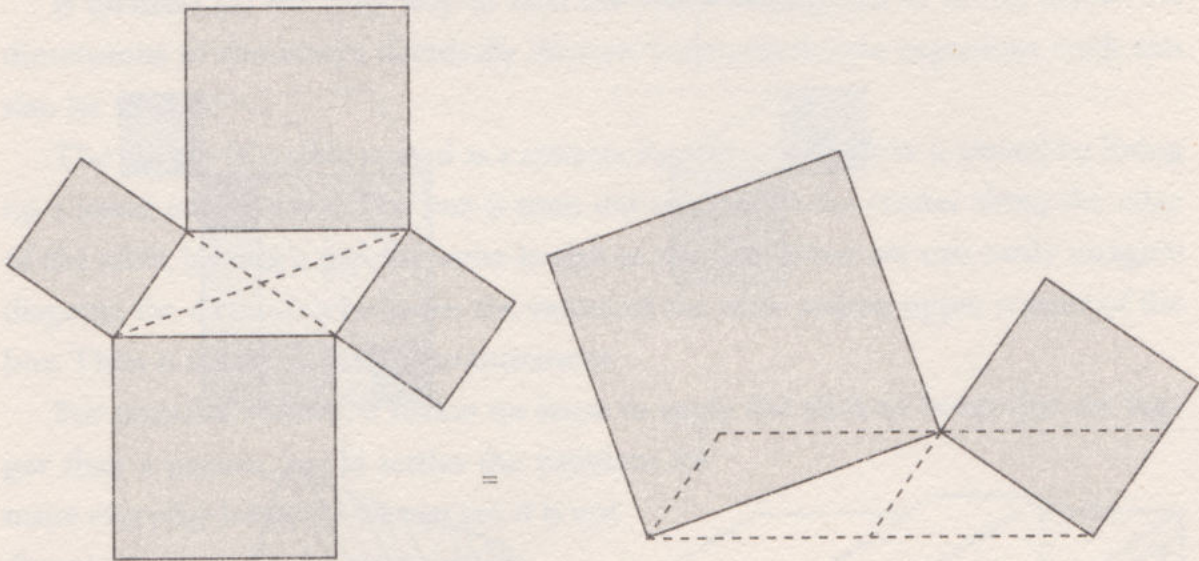
The left-hand parts have an area of $b \cdot (a \cos C)$. The two fragments also appear, $b \cdot (c \cos A)$ giving,

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos A,$$

which is known as the cosine law. When $A = 90^\circ$, $\cos 90^\circ = 0$, and we get $b^2 + c^2 = a^2$. In other words, the good old Pythagorean solution. The cosine law is an extension of the classic theorem.

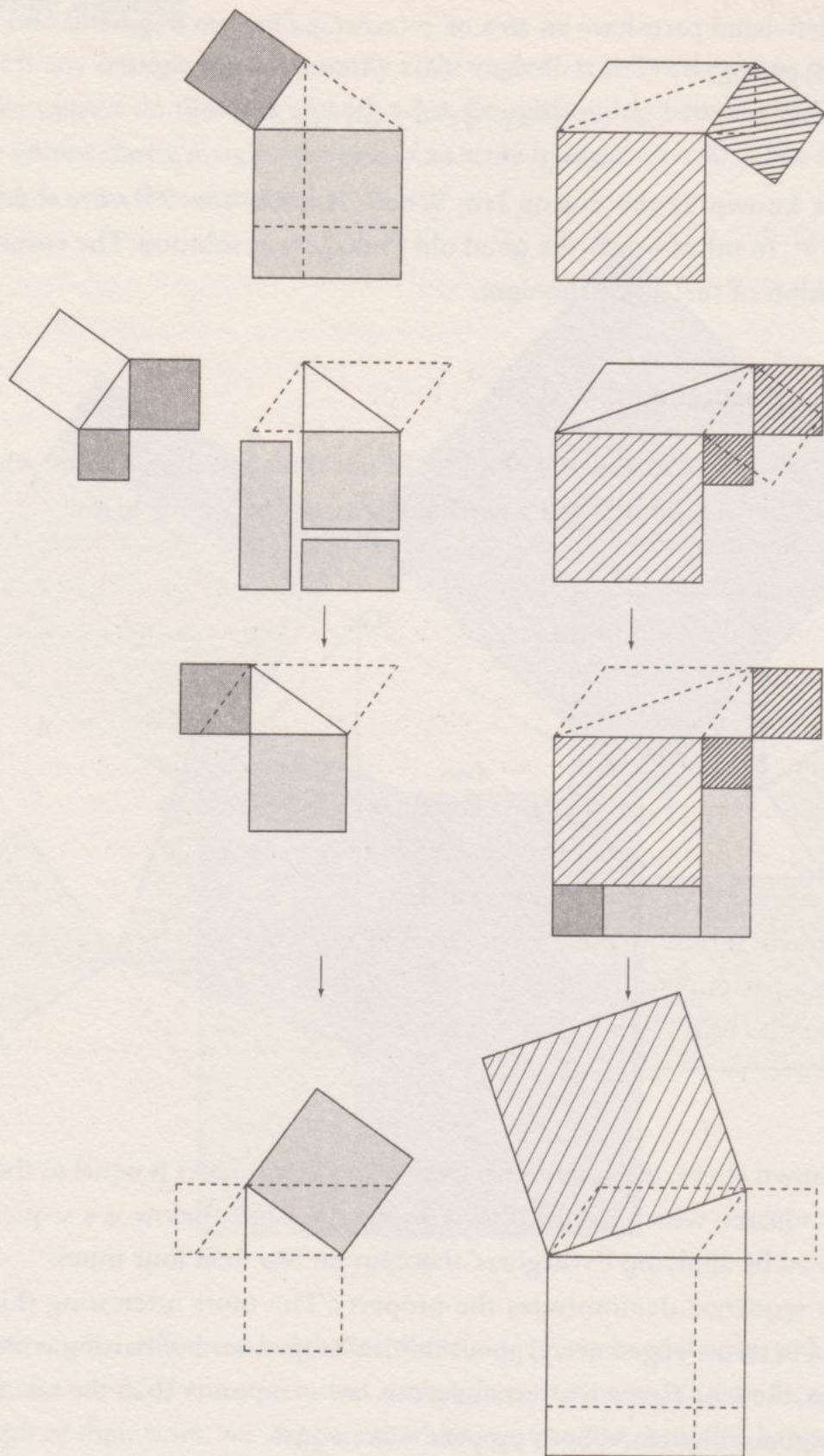
The parallelogram law

Now we present an interesting property of not three but four squares, which can be generated from the sides of a parallelogram, which we see below.



As shown above, the sum of the areas of the four squares is equal to the sum of the two squares, which can be drawn on the diagonals. Below is a sequence that is obtained by applying Pythagoras' theorem no less than four times.

This sequence demonstrates the property. The most interesting thing (and beautiful in terms of geometry) about this is that if, instead of starting with a parallelogram, the base figure is a rectangle, this law is no more than the raw theorem of Pythagoras. Therefore, both properties are equal.



The Alsina and Nelsen sequence (2006) demonstrates the law of the parallelogram, which shows that the sum of the areas of the squares drawn on the sides of a parallelogram are equal to that of the squares on its diagonals.

Pythagoras in 3D

In the first chapter we saw an immediate extension of Pythagoras' theorem in three-dimensional space. The main diagonal d of a box with edges a, b, c is given by

$$d = \sqrt{a^2 + b^2 + c^2}. \quad (3)$$

This quick exercise was just a taster that should have built up your appetite for finding out about Pythagorean ratios in 3D space.

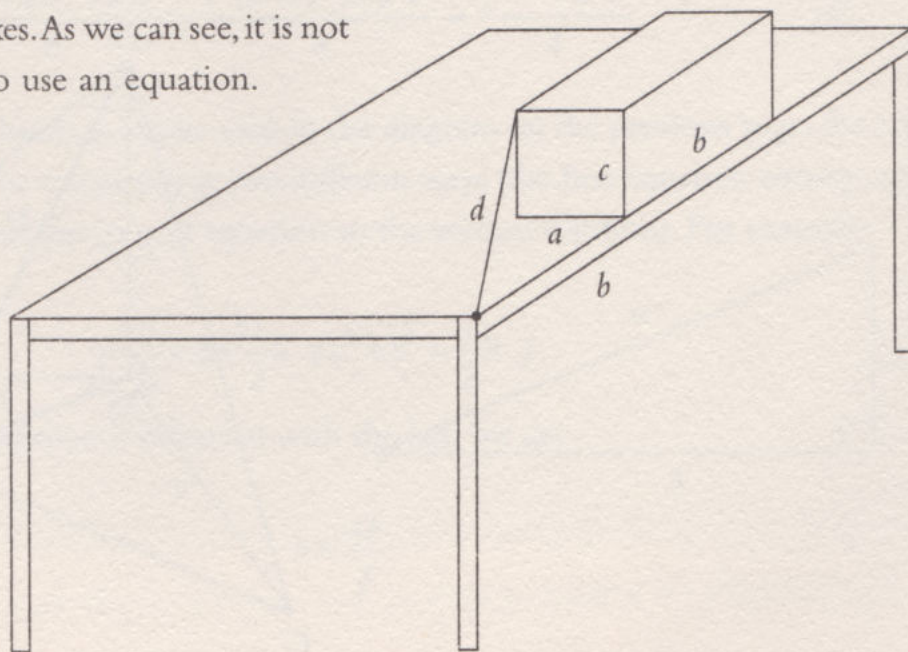
Practical measurements without Pythagoras

Imagine that you wanted to know the inside diagonal of a box that was impossible to open or had a solid interior (a wrapped present perhaps). We have to calculate the main diagonal d from edges a, b, c because we cannot measure inside it.

Is formula (3) the only way to find the value of diagonal d ? If you know the dimensions of the box, a decidedly ancient but nevertheless ingenious trick can also be used.

The corner of a table is used as a reference point, and the box is placed by lining up a lower edge with it. The box is then slid away from the corner along the edge of the table, leaving a gap the same length as the box. Now, we can easily imagine diagonal the d running between the vertex of the table and an upper corner of the box. Then it is easy to take a measurement.

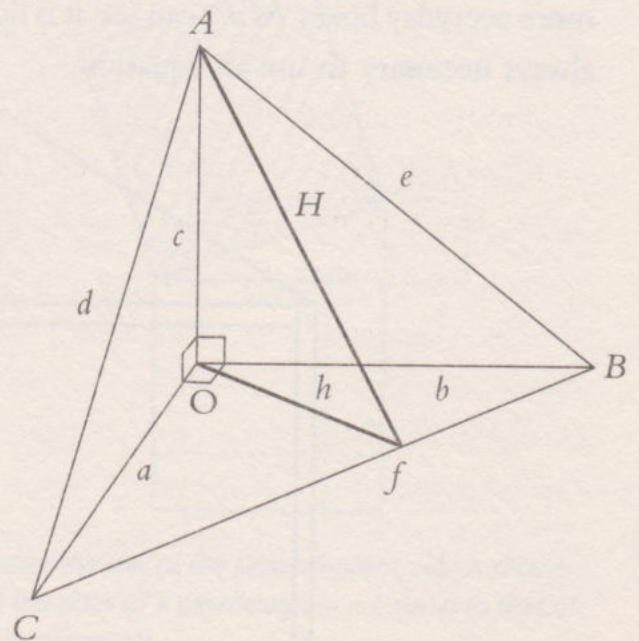
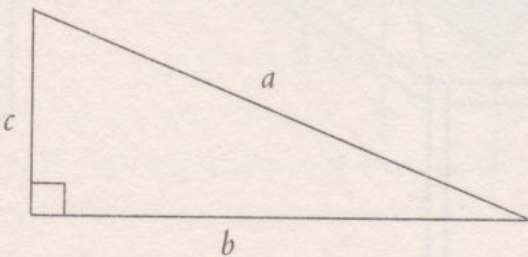
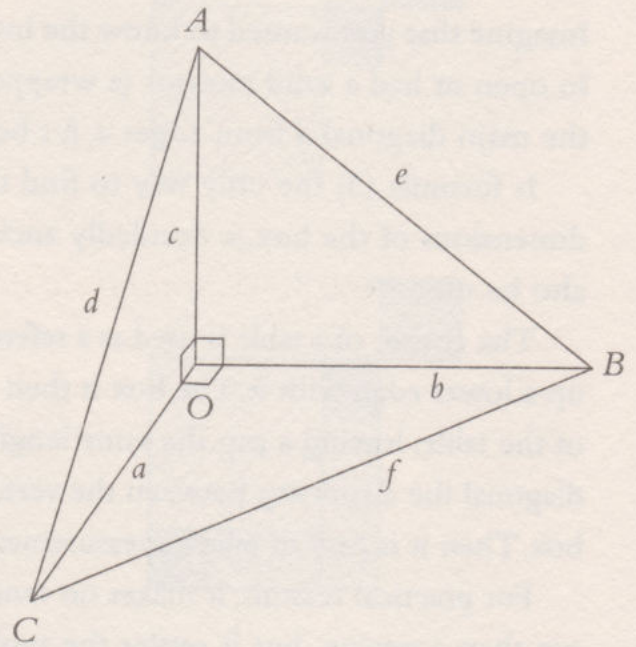
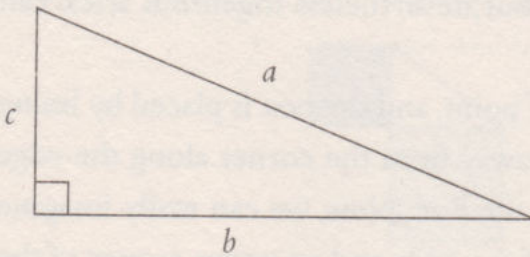
For practical reasons, it makes no sense to apply the trick to boxes that are bigger than a person, but it settles the problem for more everyday boxes. As we can see, it is not always necessary to use an equation.



Sometimes, perhaps often, cunning and ingenuity is worth more than a thousand elegant formulae.

From the right-angled triangle to the straight tetrahedron

The ease with which shapes can be transferred from a plane (two dimensions) to space (three dimensions) is well understood, but what would happen if we tried to project Pythagoras' theorem into 3D?



To transfer from plane to space, a possible option is to consider that flat triangles, determined by three vertices, are equal in 3D to tetrahedrons determined by four vertices. Following this reasoning, the right-angled triangle would have its three-dimensional translation in the straight tetrahedron, where three perpendicular edges come together at one vertex.

In this way, if Pythagoras' theorem in 2D relates the square of the hypotenuse to the squares of the other edges, in the straight tetrahedron in 3D, it is worth asking whether there is there any relationship between the square of the area of triangle ABC and the squares of the areas of the other three triangular faces. Surprisingly, the answer is yes! But that's not all, this relationship provides a perfect equivalent to the traditional Pythagorean relationship:

$$\text{Area}(ABC)^2 = \text{Area}(ACO)^2 + \text{Area}(ABO)^2 + \text{Area}(BCO)^2.$$

In trying to resolve this problem, the so-called Gua's Theorem comes up, named after its discoverer the clergyman Jean Paul de Gua de Malves, a French mathematician of *Encyclopédie* fame.

In order to verify the relationship, we start with the fact that the number of faces along the perpendicular edges of the right-angled triangles is 3. Therefore:

$$\begin{aligned} \text{Area}(ACO)^2 + \text{Area}(ABO)^2 + \text{Area}(BCO)^2 &= \left(\frac{1}{2}ac\right)^2 + \left(\frac{1}{2}bc\right)^2 + \left(\frac{1}{2}ab\right)^2 = \\ &= \frac{a^2c^2 + b^2c^2 + a^2b^2}{4} = \frac{c^2(a^2 + b^2) + a^2b^2}{4} = \frac{f^2c^2 + a^2b^2}{4}. \end{aligned} \quad (4)$$

On the other hand, as can be seen in the diagram on the previous page, the area of base COB can be calculated in two different ways (the first equation corresponds to the first way and the second equation to the second solution). For example:

$$\frac{1}{2}a \cdot b = \frac{1}{2}h \cdot \sqrt{a^2 + b^2} = \frac{1}{2}h \cdot f.$$

If the first expression is balanced with the last, we get

$$h = \frac{ab}{f}.$$

Again, by Pythagoras, the area of triangle ABC is calculated, H being its height.

$$H^2 = c^2 + h^2 = c^2 + \frac{a^2 b^2}{f^2},$$

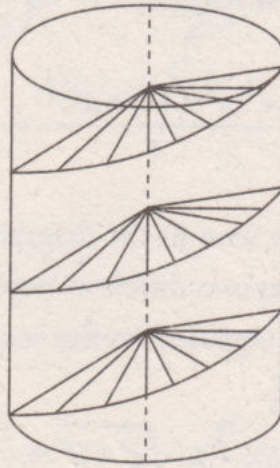
and therefore

$$\text{Area}(ABC)^2 = \left(\frac{1}{2} f \cdot H \right)^2 = \frac{1}{4} f^2 \left(c^2 + \frac{a^2 b^2}{f^2} \right) = \frac{f^2 c^2 + a^2 b^2}{4},$$

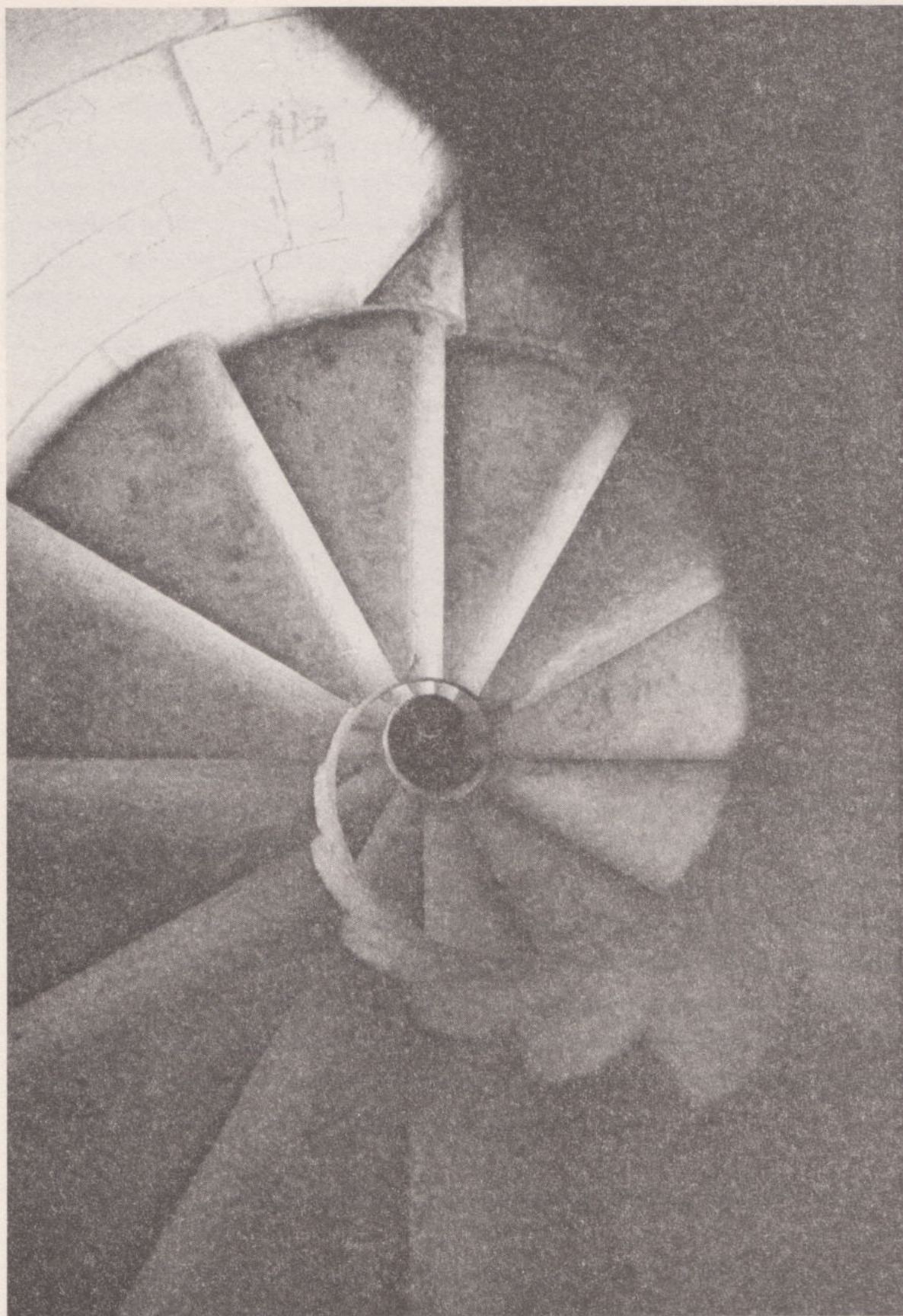
which is exactly the expression we were looking for. (4)

Pythagoras' theorem and the spiral staircase

Most people can conjure the image of a medieval tower, perhaps with a princess secluded at the very top, only accessible by a long spiral staircase. We can imagine a knight dressed in armour, who climbs the steps with his sword at the ready – he has to keep turning as he ascends but does not know what awaits around the never-ending curve of the disconcerting spiral staircase. He runs his other hand along the cold stone walls so that he does not fall. His feet find his way up the triangular steps, each one of which connects the central axis to the outer surface of the cylinder.



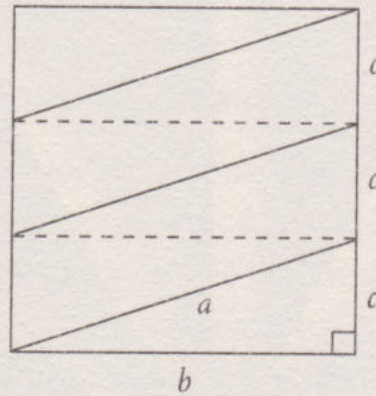
For the purposes of geometry, we can dispense with the knight for now – he should know how to take care of himself – and observe the cylindrical surface from the outside. The geometric basis of the spiral staircase is the beautiful winding curve mathematicians call a helix.



Helix-shaped spiral staircase designed by Antoni Gaudí for the Sagrada Familia in Barcelona.

THE MYSTERY OF THE PERFECT BOX

Pythagorean triples made it clear that there were infinite rectangles, the sides and diagonals of which were integers. But, is it possible to have a parallelepiped (a box) with integer edges a, b, c , such that the diagonals of the faces $\sqrt{a^2+b^2}, \sqrt{b^2+c^2}, \sqrt{a^2+c^2}$ and the main diagonal $\sqrt{a^2+b^2+c^2}$ are integers? This box mystery is yet to be solved. Polyhedrons have been found with a similar perfection, but never a simple box.

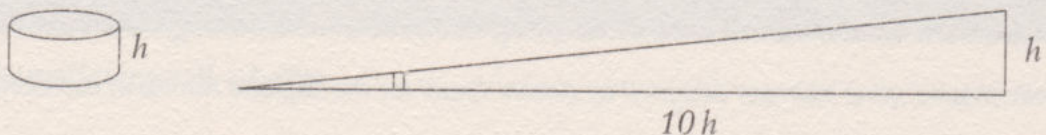


Now geometry takes the side of the evil king who has trapped the princess, and constructed a tower with a spiral staircase designed to scare knights in shining armour who get involved in situations that are none of their business. In order to build a helix with paper, all you need to do is take a rectangular piece of paper and mark the three right-angled triangles with equal and parallel hypotenuses. A helix is then formed by connecting the two opposite edges of the paper. The curve that is outlined has a constant slope like that of a spiral staircase.

In this example, Pythagoras' theorem allows us to calculate the total length L of the helix:

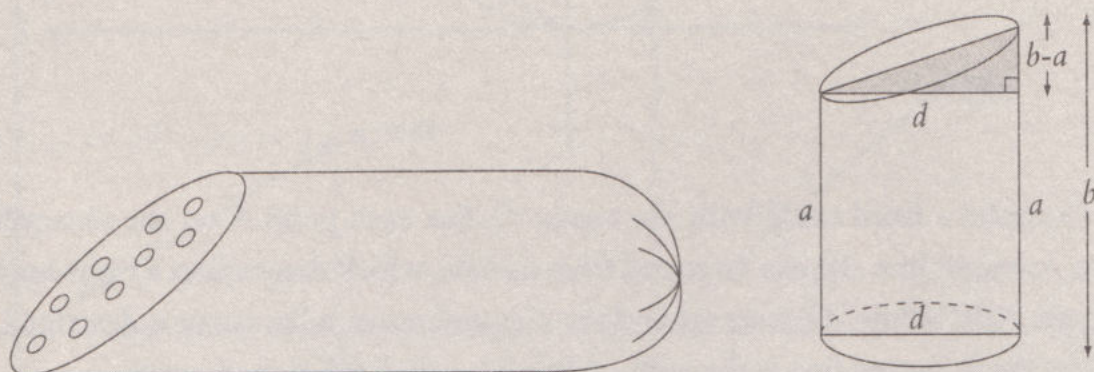
$$L = 3 \cdot a = 3 \cdot \sqrt{b^2 + c^2}.$$

Now consider another interesting case, starting with a cardboard tube of height h .



THE THEOREM OF PYTHAGORAS AND SLICING SALAMI

Salami is cylinder-shape. Cutting it at an angle with a knife produces an elliptical shape. As can be seen in the diagram, if d is the diameter of the salami, the size of the ellipse L depends on the size of the angle as determined by the length $b-a$. Using Pythagoras' theorem, L will be $L = \sqrt{(b-a)^2 + d^2}$. Thus, d would never vary, but a cut can be made with large slices (only limited by the total length of the salami). Controlling the size of these slices depends on the Pythagorean ratio, but its thickness will depend more on how hungry we are!



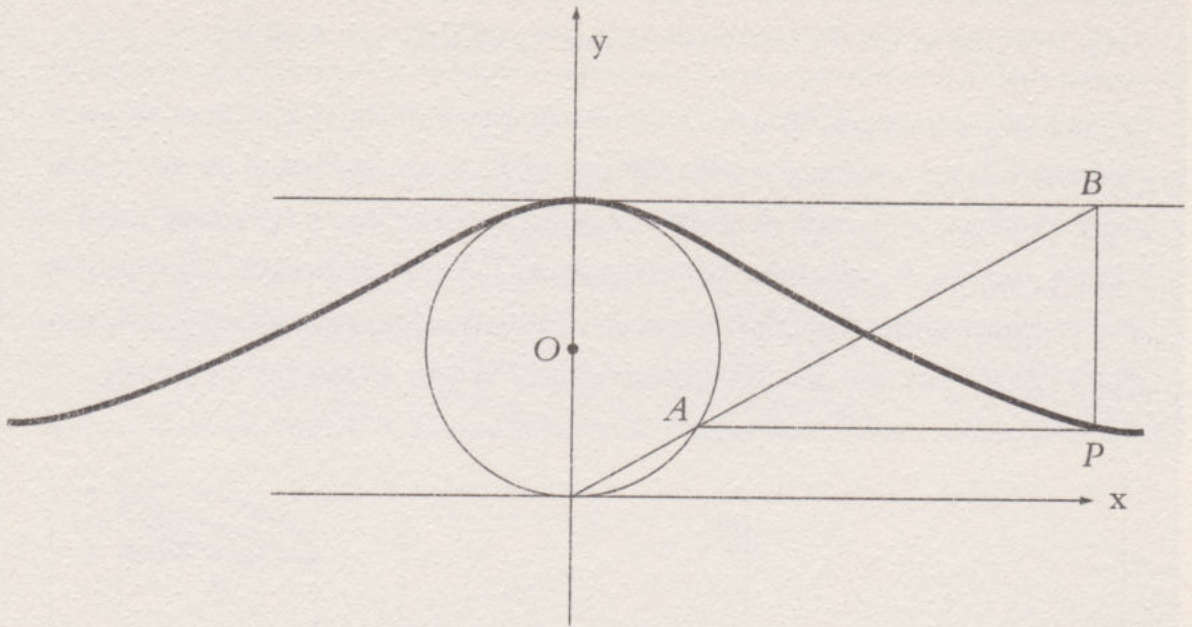
We want to draw a helix with a 10% slope on it and calculate its length. How should we go about doing this? On paper, construct a right-angled triangle with edges h and $10h$. With these dimensions, the slope of the hypotenuse is

$$\frac{\text{opposite cathetus}}{\text{adjacent cathetus}} = \frac{h}{10h} = \frac{1}{10} = 0,1,$$

or 10%. The value of the hypotenuse will be $\sqrt{h^2 + (10h)^2} = h\sqrt{101}$, and, by rolling the triangle of paper around the cylinder, we get the helix we were looking for (with the number of turns more or less equalling the value of r).

The Agnesi curve

The great Italian mathematician María Gaetana Agnesi (1718–1799) made numerous contributions to the development of mathematics. Among them is the curve that carries her name, which is based on a Pythagorean property.



Consider a fixed circle with the centre O . For each point A on the circumference, a straight line AB can be traced from its base, which determines a right-angled triangle ABP . While A moves around the circumference, what curve is described by P ? The thick curved line is the path of P and is called the Agnesi curve.

Its equation is $y = \frac{1}{1+x^2}$ and the area under the curve is π .

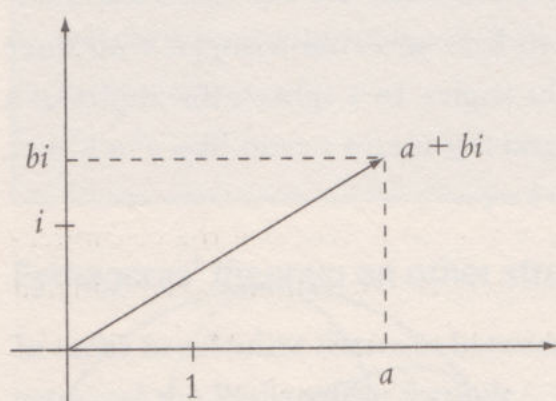
Finally, imaginary numbers

The study of square roots and their use in resolving quadratic equations threw up mysterious expressions such as $\sqrt{81-144}$, which do not make any real sense. Likewise $x^2 = 1$ had ± 1 as solutions but $x^2 = -1$ did not have real solutions.

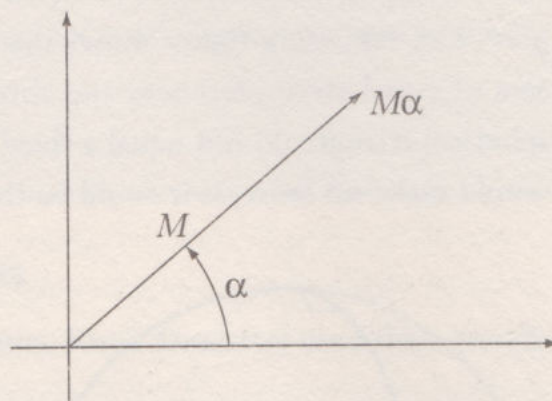
THE WITCH OF AGNESI

In the English-speaking world, the Agnesi curve is known as the 'Witch of Agnesi' due to a bad translation. In Italian it is given the name *versiera*, a naval term that identifies the rope used to turn a sail, which is very similar to the shape drawn by the curve. A poor translator confused the word with *avversiera*, which means 'she-devil'.

At some point, Renaissance mathematician and philosopher Girolamo Cardano proposed the problem of breaking down the number $10 = x + y$ in such a way that its product $x \cdot y = 40$. From there he came up with the expressions $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$. Many years were to pass until the great names of mathematics from the 18th and 19th centuries, Euler, Wessel, Argand and Gauss, concluded the creation of complex numbers, an extension to real numbers where all polynomial equations have a solution. With the symbol $i = \sqrt{-1}$ complex numbers are expressions like $a + bi$, a being the real part and b the imaginary part.



Complex number in binomial form



Complex number in polar form

The key to manipulating these expressions was to achieve a graphical representation of the complex numbers, treating each $a + bi$ as a vector with real coordinate a on the horizontal axis and imaginary coordinate bi on the vertical axis. It is at this point that Pythagoras' theorem comes up again. The vector $a + bi$ could also be identified by means of two parameters M_α , where $M = \sqrt{a^2 + b^2}$ is the module, or absolute value of the number $a + bi$, and angle is the argument, or angle that forms the vector with the horizontal real axis $\tan \alpha = b/a$. Thanks to this Pythagorean representation the multiplication of complex numbers

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

was born, if M_α corresponded to $a + bi$ and N_β corresponded to $c + di$

$$M_\alpha \cdot N_\beta = (M \cdot N)_{\alpha+\beta}.$$

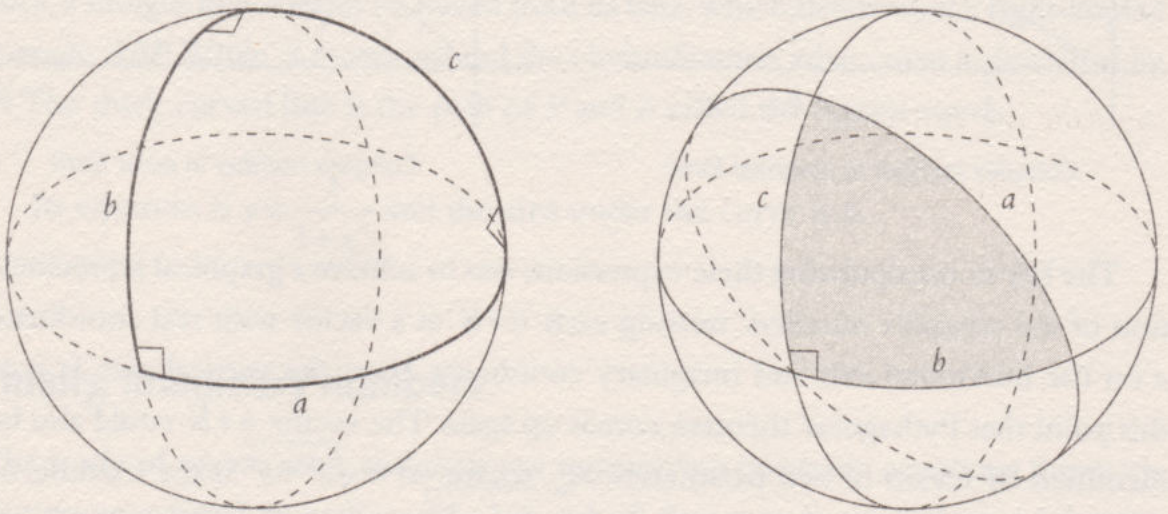
Complex numbers are essential in algebra or complex analysis and have interesting applications in electrical engineering, electromagnetism, chaos theory, quantum physics, the theory of fractals, control theory, signal theory and countless other mathematical branches, both pure and applied.

The omnipresent theory

You have seen Pythagoras' theorem in two dimensions and been able to appreciate its projection into three-dimensions. Even though it may seem that these are the only two possible spaces, mathematics make many more possible.

Pythagoras' theorem on other surfaces

Does it make sense to look for Pythagorean relationships on the surface of a sphere? If straight lines are drawn on a sphere, they might follow the circumference (along meridians, equator, etc.). Therefore, there are also 'spherical triangles'. The following figure shows that we can have spherical triangles with three arcs of equal circumferences and three right angles. In a sphere, the angles of a spherical triangle do not equal a fixed sum and this means a ratio like $a^2 = b^2 + c^2$, would make no sense, as it could be that $a = b = c$.



We could of course consider spherical triangles, and the relationship between the sides a, b, c , could be investigated, taking into account the radius of the sphere R and the cosines of the angles from the centre of the sphere. However, the Pythagorean magic simply falls down in this case.

One ingenious way of achieving the proposed goal – to a certain point – is to start with a sheet of paper where there is a right-angled triangle for which Pythagoras' theorem works, and use it to form a curved surface. If we made a cylinder, the straight-line triangle would become curved. Here we have a demonstration of how important the imagination is in mathematics.

LITERARY HORRORS WITH PYTHAGORAS

Literary minds have taken great advantage of the words associated with Pythagoras, such as hypotenuse. All too often though, to investigate the theorem's appearances in fiction is to fall into a pit of literary horrors. For example, in the celebrated movie of *The Wizard of Oz*, one of the characters recites the theory as "the sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the other side..." The statement confuses isosceles with 'right-angled' – the two definitions rarely coincide. However, in far less literary, but more fun terms, in the story *The Crime of the Hypotenuse*, which appears in the book *The Mansions of the Gods* from *The Adventures of Asterix and Obelix*, one of Julius Caesar's architects is very aptly named Squaronthehypotenus.

Pythagoras' theorem on other structures

In order to calculate distances between vectors, René Descartes used their coordinates and the Pythagorean formula.

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

This formula can be extended to vectors with any number n of coordinates. Thus, for $n = 4$, we would get a Pythagorean version in four-dimensional space.

Analytic or Cartesian geometry was born by considering the points on a plane or in a 3D space using vectors and coordinates. With this language, Descartes fused classic geometry with arithmetic. The French philosopher took Euclid's geometry into account but complemented it with corresponding numeric measurements. Descartes himself said in a letter to princess Elizabeth of Bohemia:

"...I do not consider any other theorems but that the sides of similar triangles have a similar proportion between them, and that in right triangles, the square of the base is equal to the sum of the square of the sides. I do not fear supposing more unknown quantities to reduce the problem to such terms that it depends only on these two theorems."

In structures known as normed vector spaces, the norm, module or length of the vector $\vec{x}$ is represented by $\|\vec{x}\|$, and is very useful when considering orthogonal relationships – where the vectors are perpendicular:

$$\vec{x} \text{ is orthogonal to } \vec{y} \text{ if } \|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2. \quad (5)$$

Here the Pythagorean relationship no longer relates to areas of squares but now determines orthogonality. The ‘vectors’ of these structures can be more abstract elements than the arrows of a plane or space – for example, continuous functions f defined by the interval $[0,1]$, with real values.

Its norm can be calculated with the expression

$$\|f\| = \left(\int_0^1 f(x)^2 dx \right)^{1/2} \quad \text{or} \quad \|f\| = \text{maximum } \{|f(x)|, x \text{ in } [0,1]\}.$$

That is how equations like (5) would be obtained expressing orthogonality between functions, sequences or any elements of the ‘vectorial structure’.

In so-called Hilbert spaces, the scalar product $\vec{x} \cdot \vec{y}$ of vectors is considered, for example, on a plane

$$(x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2.$$

And the norms or lengths of vectors are calculated by means of the expression $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$. So the orthogonality of vectors can be confirmed from the property $\vec{x} \cdot \vec{y} = 0$, which is the equivalent to Pythagoras’ theorem:

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 \quad \text{if, and only if } \vec{x} \cdot \vec{y} = 0.$$

As can be seen, the Pythagorean relationship does not just form the most endearing and mysterious part of the history of mathematics. It is very much at the forefront of mathematics.

Epilogue

Pythagoras' theorem is not only an essential mathematical result for discovering right-angled triangles with very useful applications in many fields. If it has lasted through time and even today continues to be relevant, it is because it represents the nucleus of the ambitious Pythagorean philosophical and mathematical project, a way of thinking based on the powerful idea that it is possible to contemplate the world and understand it through numbers.

Scientist John Bernal was a pioneer in X-ray crystallography, but also a respected non-fiction author, producing works such as the influential *Science in History*. Regarding Pythagoras, he states: "...the truth is that the relationship established by his teachings, between mathematics, science and philosophy, has never been forgotten." His opinion is shared and expressed much more emphatically by the prestigious philosopher, logician and mathematician Bertrand Russell, in his *History of Western Philosophy*: "I do not know of any other man who has been as influential as he was in the sphere of thought. I say this because what appears as Platonism is, when analysed, found to be in essence Pythagoreanism." To which Russell himself later added, in one of his habitually far-sighted and penetrating observations: "The strangest thing about modern science is its regression to Pythagoreanism."

At the beginning of the 21st century, the digital revolution had finished its opening act and had begun a consolidated transformation of the world. Currently, the idea that numbers form the foundation of all manifestations of science, technology, and therefore, every-day life is harder to dispute than ever. Pythagoreanism in its pure state flows through all formulae and calculations that describe the behaviour of nature and its physical and chemical phenomena; in genetics, in clinical analyses and in microsurgery; in climate change predictions and in sociological analysis; in economic arguments and those of social science; in musical scores, in dance and in every one of the verses that we read in search of something beautiful.

The theorem of Pythagoras is, in a way, the theorem of our lives, because, today, more than ever, our lives are Pythagorean.

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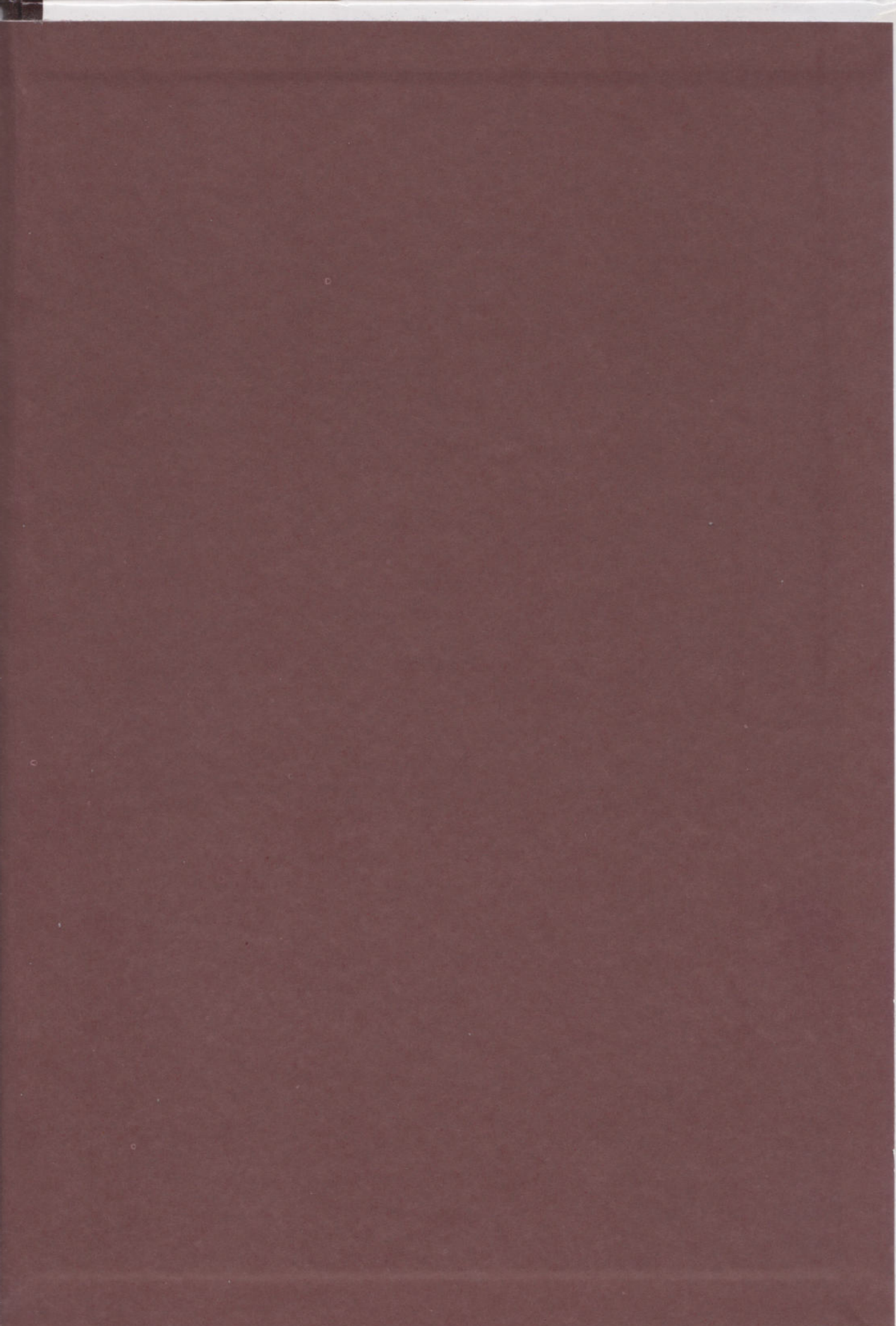
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Pythagoras' Theorem

The sacred geometry of triangles

The relationship between the hypotenuse and the adjacent sides of a right-angled triangle is one of the most important scientific discoveries in the history of humanity. The relationship has some surprising effects both in geometry and in the theory of number. The theorem that describes this relationship takes its name from Pythagoras, one of the most intriguing and mysterious figures in the history of science. The first mathematical superstar was also the leader of a mystical circle who regarded mathematics as a religion that would explain everything in the Universe.